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Finite element solution of the Navier–Stokes equations

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CONTENTS

1. Introduction

Viscous incompressible flows are of considerable interest for applications. Let us mention, for example, the design of hydraulic turbines or rheologically complex flows appearing in many processes involving plastics or molten metals. Their simulation raises a number of difficulties, some of which are likely to remain while others are now resolved. Among the latter are those related to incompressibility which are also present in the simulation of incompressible or nearly incompressible elastic materials. Among the still unresolved are those associated with high Reynolds numbers which are also met in compressible flows. They involve the formation of boundary layers and turbulence, an ever present phenomenon in fluid mechanics, implying that we have to simulate unsteady, highly unstable phenomena.

This article will deal mainly with problems associated with incompressibility effects but will also try to address the other issues. It will not be

an exhaustive presentation and will evidently be somewhat biased by the prejudices of the author and his ignorance of many areas of an ever growing literature. The reader might consult the books by Girault and Raviart (1986) where other mathematical aspects of the problem are treated. The book by Hughes (1987) is application-oriented and is a good reference for those interested in the actual implementation of finite element methods. The reader should also refer to Pironneau (1989) or to Thomasset (1981) for more information and other aspects of the problem.

2. The finite element method

2.1. Sobolev spaces

Let $L^2(\Omega)$ be the space of square integrable functions. We then define the Sobolev spaces,

$$H^m(\Omega) = \{v \mid v \in L^2(\Omega),\ D^\alpha v \in L^2(\Omega),\ |\alpha| \leq m\} \tag{2.1}$$

where $D^\alpha v = \partial^{|\alpha|} v / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$. For our purpose,the most important of these spaces will be $H^1(\Omega)$ (and some of its subspaces). We define on $H^m(\Omega)$, the semi-norm

$$|v|_{m,\Omega} = \Big(\sum_{|\alpha|=m} \int_\Omega |D^\alpha v(x)|^2 \underset{\cdot}{x} \Big)^{1/2}. \tag{2.2}$$

It is then clear that $|v|_{0,\Omega}$ is the usual norm on $L^2(\Omega)$. In general, we shall use on $H^m(\Omega)$ the standard norm

$$\|v\|_{m,\Omega} = \Big(\sum_{|\alpha|\leq m} |v|^2_{m,\Omega} \Big)^{1/2}, \tag{2.3}$$

which on $H_0^1(\Omega)$ reduces to $\|v\|_{1,\Omega} = \Big(|v|^2_{0,\Omega} + |v|^2_{1,\Omega}\Big)^{1/2}$.

2.2. Conforming finite elements

We shall be interested here in *finite element approximations* of $H_0^1(\Omega)$ and $L^2(\Omega)$. It is not possible to give a complete presentation of the finite element methods as this would require a book in itself. We refer to Ciarlet (1978), Ciarlet and Lions (1991), Hughes (1987), Raviart and Thomas (1983) or to the classical Zienkiewicz (1977) for a general presentation. For more specific issues and details on many of the topics introduced here, Brezzi and Fortin (1991) should be a suitable reference. We nevertheless need a minimum of notation.

The basic idea of the finite element method is to construct a *partition* $\mathcal{T}_h$ of the domain Ω by subdividing it into triangles or quadrilaterals which will be called elements. One then builds approximations using polynomial functions

defined element-wise with some continuity requirements at the interfaces between the elements. To approximate $L^2(\Omega)$ no continuity is required while $C^0(\Omega)$-continuity yields correct approximations of $H^1(\Omega)$. Higher-order continuity properties would be required for higher-order Sobolev spaces. To describe finite element approximations more precisely we shall need a few definitions. Let us define on an element $K \in \mathcal{T}_h$ the space of polynomials of degree $\leq k$,

$$P_k(K) = \Big\{ p(x_1, x_2) \mid p(x_1, x_2) = \sum_{i+j \leq k} a_{ij}\, x_1^i\, x_2^j \Big\}. \tag{2.4}$$

The dimension of $P_k(K)$ is $(k+1)(k+2)/2$ for $n = 2$ and for $n = 3$, $(k+1)(k+2)(k+3)/6$. We shall also use, (for $n = 2$)

$$P_{k_1,k_2}(K) = \Big\{ p(x_1, x_2) \mid p(x_1, x_2) = \sum_{\substack{i \leq k_1 \\ j \leq k_2}} a_{ij}\, x_1^i\, x_2^j \Big\} \tag{2.5}$$

the space of polynomials of degree $\leq k_1$ in x_1 and $\leq k_2$ in x_2. In the same way we can define $P_{k_1,k_2,k_3}(K)$ for $n = 3$. The dimension of these spaces is respectively $(k_1+1)(k_2+1)$ and $(k_1+1)(k_2+1)(k_3+1)$. We then define

$$Q_k(K) = \begin{cases} P_{k,k}(K) & \text{for } n = 2, \\ P_{k,k,k,}(K) & \text{for } n = 3. \end{cases} \tag{2.6}$$

The classical finite element approximations are obtained by using polynomials like $P_k(\hat{K})$ or $Q_k(\hat{K})$ on some *reference element* $\hat{K}$ and to carry them over to an arbitrary element K by a change of variable:

$$v_h|_K = \hat{v} \circ F^{-1}, \tag{2.7}$$

where $K = F(\hat{K})$ and $\hat{v}$ is a polynomial function on $\hat{K}$. Continuity is obtained by a suitable choice of the degrees of freedom, that is, interpolation points defining the polynomials. The simplest case is described in the following example.

Example 2.1 (Affine finite elements.) This is the most classical family of finite elements. The reference element is the triangle $\hat{K}$ of Figure 2.1 and we use the affine transformation

$$F(\hat{x}) = x_0 + B\hat{x}\,, \tag{2.8}$$

where B is a two-by-two matrix.

The element $K = F(\hat{K})$ is an arbitrary triangle and it is not degenerate provided $\det B \neq 0$. We now take $\hat{P} = P_k(\hat{K})$ and choose an appropriate set of degrees of freedom. The standard choices for $k \leq 3$ are presented on Figure 2.2 where the dots represent the degrees of freedom. One notes that this choice of points ensures continuity at the element interfaces. □

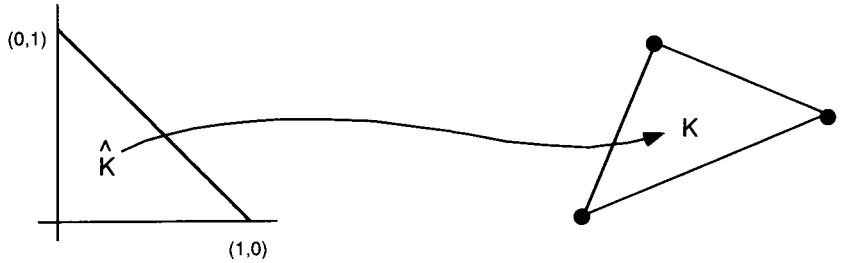

Fig. 2.1. Affine transformation.

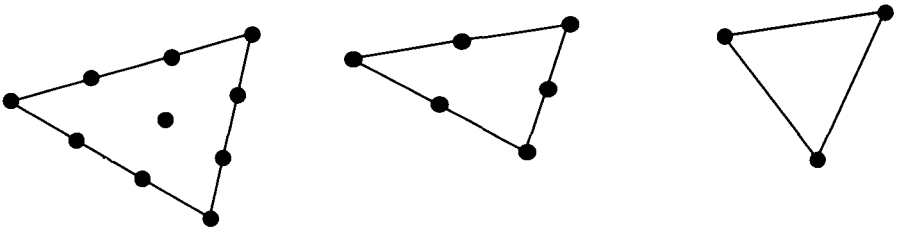

Fig. 2.2. Standard affine elements.

The next example presents a second classical family of finite elements. They are defined on arbitrary quadrilaterals and will not be polynomial functions even though they are obtained by applying a change of variables to a polynomial.

Example 2.2 (Quadrilateral elements.) The reference element concerned is taken to be the square $\hat{K} =]0,1[\times]0,1[$. We take $\hat{P} = Q_k(\hat{K})$ and a transformation F with each component in $Q_1(\hat{K})$. We present the standard choice of degrees of freedom for $k = 1$ in Figure 2.3. It must be noted that we need $F \in (Q_1(\hat{K}))^2$ to define a general straight-sided quadrilateral. □

Finally we recall that it is possible to employ curved elements to obtain better approximation properties near the boundary of the domain Ω.

Example 2.3 (Isoparametric elements.) Let us first consider the triangular case. We shall use the same reference element and the same set $\hat{P}$ as in Example 2.1. We now take the transformation $F(\hat{x})$ so that each of

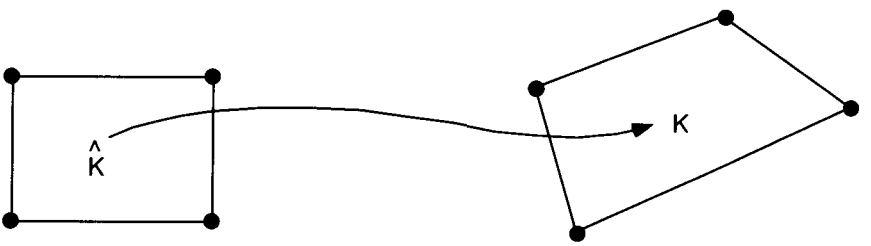

Fig. 2.3. Q_1 isoparametric element.

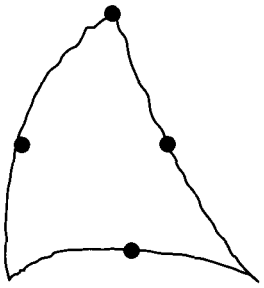

Fig. 2.4. Curved triangular element.

its components F_i belongs to $P_k(\hat{K})$. For $k = 1$ nothing is changed but for $k \geq 2$, the element K now has curved boundaries. We depict the case $k = 2$ in Figure 2.4. It must be noted that the curvature of element boundaries introduces additional terms in the approximation error and the curved elements should be used only when they are really necessary (Ciarlet and Raviart (1972) or Ciarlet (1978)). Similar constructions enable us to define isoparametric quadrilateral elements using $F \in Q_k(K)$. □

We note again that the degrees of freedom have been chosen in order to ensure continuity between elements. We also need some basic results about the accuracy of interpolation by finite element functions. In dealing with $H^1(\Omega)$, one generally employs Lagrange interpolation, that is, the value of the interpolant at the degrees of freedom are computed from the value of the functions at these points, excluding derivatives. There is, however, a difficulty as point values of functions in $H^1(\Omega)$ are not, in general, defined. This can be circumvented by using the technique of Clément (1975) where local averages are employed instead of point values. The details are beyond the scope of this article. We shall only cite a very basic result, assuming r_h to be defined by the usual Lagrange interpolant.

Proposition 2.1 If the mapping F is affine, that is $F(\hat{x}) = x_0 + B\hat{x}$, and if $r_h p_k = p_k$ for any $p_k \in P_k(K)$, we have for $v \in H^s(\Omega)$, $m \leq s$, $1 < s \leq k+1$

$$|v - r_h v|_{m,K} \leq c \, \|B^{-1}\|^m \, \|B\|^s \, |v|_{s,K}. \tag{2.9}$$

As we said above, the condition $s > 1$ is required in order to ensure that point values of the function to be interpolated are well defined and the result can be improved (cf. Brezzi and Fortin (1991)). To obtain global results on Ω, we shall need some assumption to ensure that the partition $\mathcal{T}_h$ is not degenerate, i.e. that the angles of the triangles are bounded away from π.

Let then h_K be the diameter of K, and let us define, for affine elements,

$$\sigma_K = \frac{h_K}{\rho_K} \tag{2.10}$$

where ρ_K is the diameter of the largest inscribed disk (or sphere) in K. We then say that a family of triangulations $(\mathcal{T}_h)_{h\geq 0}$ is regular if

$$\sigma_K < \sigma, \quad \forall K \in \mathcal{T}_h, \ \forall h. \tag{2.11}$$

One may then prove the following result.

Proposition 2.2 If $(\mathcal{T}_h)_{h\geq 0}$ is a regular family of affine partitions, there exists a constant c depending on k and on σ and an interpolation operator Π_1 such that

$$\sum_K h_K^{2m-2}|v - \Pi_1 v|^2_{m,K} \leq c\|v\|^2_{1,\Omega} \, m = 0, 1. \ \square \tag{2.12}$$

For more general partitions including general isoparametric elements, the result is qualitatively the same: we have an $\mathcal{O}(h^k)$ approximation provided the family of partitions is regular in a sense which has to be made precise for each type of partition.

Finally, we introduce some notation for the usual spaces of finite element approximations. We thus define

$$\mathcal{L}^s_k = \{v \mid v|_K \in P_k(K), \ v \in H^s(\Omega)\}. \tag{2.13}$$

In the same way we shall write $\mathcal{L}^s_{[k]}$ when $\mathcal{T}_h$ consists of quadrilaterals and the local approximations are built from $Q_k(\hat{K})$ by an appropriate change of variables. We shall also quite often need a class of functions called *bubble functions*. For an element K a bubble function is a function vanishing on ∂K. In particular, we shall denote

$$\begin{cases} B_k = (P_k(K) \cap H^1_0(K)), \\ B_{[k]} = Q_k(K) \cap H^1_0(K)). \end{cases} \tag{2.14}$$

2.3. Scaling argument

In some of the proofs, we shall invoke scaling arguments in order to express the dependence of some quantities on the finenes of the mesh. The standard procedure (cf. Ciarlet (1978)) is to map the quantity to be estimated on a reference element $\hat{K}$ on which it can be computed and then to study the effect of the change of variables which maps $\hat{K}$ to an arbitrary element K in the partition. An interesting variant of this procedure, introduced by Dupont and Scott (1980), consists essentially of separating the two issues of the size and the shape of the element. Indeed, using the change of variables

$$\boldsymbol{x} = h_K \, \hat{\boldsymbol{x}} + \boldsymbol{b}, \tag{2.15}$$

one can easily check the effect of mesh size on a given quantity. In this way one sees that one has

$$\begin{cases} |v_h|_{1,K} = |\hat{v}_h|_{1,\hat{K}}\,, \\ |v_h|_{0,K} = h_K\,|\hat{v}_h|_{0,\hat{K}}\,, \end{cases} \tag{2.16}$$

and many other similar relations. The effect of shape is then treated by an argument of compactness: a continuous function is bounded on a compact set. One obtains in this way, for a general transformation,

$$|v_h|_{1,K} = c(k,\theta_0)|\hat{v}_h|_{1,\hat{K}}\,, \tag{2.17}$$

where k is the degree of polynomials employed and θ_0 is the smallest angle of the mesh. We refer the reader to Dupont and Scott(1980) or Brezzi and Fortin(1991) for more details.

3. Presentation of the problem

Let Ω be a domain of $\mathbf{R}^2$ or $\mathbf{R}^3$ and let us denote Γ its boundary. We shall want to solve in this domain, over a time interval $]0,T[$, the Navier–Stokes equations of incompressible fluid flow with initial conditions and boundary conditions. Let ρ be the density of the fluid, $\boldsymbol{u}$ its velocity and p, its pressure. We thus have to find in Ω, a solution of

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\mathbf{grad}\,\boldsymbol{u}\right) - 2\mu\,\boldsymbol{Au} + \mathbf{grad}\,p = \rho\boldsymbol{f}, \tag{3.1}$$

$$\operatorname{div}\boldsymbol{u} = 0, \tag{3.2}$$

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x). \tag{3.3}$$

In equation (3.1), we have denoted

$$\boldsymbol{Au} = \begin{cases} \dfrac{\partial^2 u_1}{\partial x_1^2} + \frac{1}{2}\dfrac{\partial}{\partial x_2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) \\ \dfrac{\partial^2 u_2}{\partial x_2^2} + \frac{1}{2}\dfrac{\partial}{\partial x_1}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right). \end{cases} \tag{3.4}$$

Taking (3.2) into account, it is easily seen that we have

$$2\boldsymbol{Au} = \triangle\boldsymbol{u}. \tag{3.5}$$

However, the variational formulation and *natural boundary conditions* will, as we shall see later, be different for these two forms of the equations. We consider a part Γ_D of Γ on which Dirichlet boundary conditions are given,

$$\boldsymbol{u}|_{\Gamma_D} = 0, \tag{3.6}$$

and a part Γ_N on which Neumann type conditions are specified, that is, in the present case, a condition on stresses is given. Let $\boldsymbol{n}$ be the outward unit normal to Γ_n and $\boldsymbol{t}$ the associated tangent vector for a two-dimensional

problem or let $\boldsymbol{t}_1, \boldsymbol{t}_2$ be tangent vectors for a three-dimensional problem. We then impose, $\boldsymbol{g} = \{g_{\boldsymbol{n}}, g_{\boldsymbol{t}_1}, g_{\boldsymbol{t}_2}\}$ being given, the boundary conditions on Γ_N

$$-p + 2\mu \frac{\partial \boldsymbol{u}\cdot\boldsymbol{n}}{\partial \boldsymbol{n}} = g_{\boldsymbol{n}}, \tag{3.7}$$

$$\mu\left(\frac{\partial \boldsymbol{u}\cdot\boldsymbol{n}}{\partial \boldsymbol{t}_i} + \frac{\partial \boldsymbol{u}\cdot\boldsymbol{t}_i}{\partial \boldsymbol{n}}\right) = g_{\boldsymbol{t}_i}, \quad i = 1, 2. \tag{3.8}$$

For two-dimensional problems, we have only one tangent vector and one condition in (3.8) instead of two. We shall, in fact, work with a variational formulation of the Navier–Stokes equations, and for this we shall need to define appropriate function spaces. Let us denote

$$V = (H_D^1(\Omega))^2 = \left\{\boldsymbol{v} | \boldsymbol{v} \in (H^1(\Omega))^2, \boldsymbol{v}|_{\Gamma_D} = 0\right\} \tag{3.9}$$

$$Q = L^2(\Omega). \tag{3.10}$$

We also define the rate-of-strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$ by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{3.11}$$

Let $\Sigma =]0, T[\times\Omega$ and let us seek a weak solution $\boldsymbol{u} \in L^2(0, T; V)$, $p \in L^2(\Sigma)$ of equations (3.1)–(3.3), that is, let us look for $\{\boldsymbol{u}, p\}$, solution of

$$\begin{cases} \displaystyle\int_\Omega \frac{\partial \boldsymbol{u}}{\partial t}\cdot\boldsymbol{v}\,\mathrm{d}x + 2\mu\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x + \int_\Omega \boldsymbol{u}\cdot\mathbf{grad}\,\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x \\ \qquad\qquad - \displaystyle\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x - \int_\Omega p\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x = 0, \quad \forall \boldsymbol{v} \in V \\ \displaystyle\int_\Omega q\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{d}x = 0, \quad \forall q \in Q, \end{cases} \tag{3.12}$$

where the meaning of $\partial\boldsymbol{u}/\partial t$ would have to be made precise.

Remark 3.1 It can be easily checked through an integration by parts that the natural boundary conditions associated with this variational formulation are precisely (3.6)–(3.8). This would not be the case had we employed, instead of

$$2\mu\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x,$$

a different bilinear form such as $\mu\int_\Omega \mathbf{grad}\,\boldsymbol{u} : \mathbf{grad}\,\boldsymbol{v}\,\mathrm{d}x$ which leads to the same equations inside Ω but with the boundary conditions

$$-p + \mu\frac{\partial \boldsymbol{u}\cdot\boldsymbol{n}}{\partial \boldsymbol{n}} = g_{\boldsymbol{n}} \text{ on } \Gamma_N, \tag{3.13}$$

$$\mu\frac{\partial \boldsymbol{u}\cdot\boldsymbol{t}}{\partial \boldsymbol{n}} = g_{\boldsymbol{t}} \text{ on } \Gamma_N. \tag{3.14}$$

It is also possible to obtain as a natural condition

$$\operatorname{rot} \boldsymbol{u}|_\Gamma = g, \tag{3.15}$$

using $\mu \int_\Omega \operatorname{rot} \boldsymbol{u} : \operatorname{rot} \boldsymbol{v}\, dx$ as a bilinear form, which still generates the same differential operator in Ω. □

We refer the reader to Temam (1977) or Lions (1969) for a complete presentation of existence and uniqueness results. One striking point with respect to these equations is the absence of an equation containing $\partial p/\partial t$: thus our system is not of the Cauchy–Kowalevska type. In fact the pressure appears here as a *Lagrange multiplier* associated with the divergence-free condition $\operatorname{div} \boldsymbol{u} = 0$. To understand this we shall, in the next section, consider the simplified steady-state Stokes problem, valid for low-speed or highly viscous flows. For the moment, we shall highlight an additional property of the above equations. They are equations of 'convection–diffusion' type, by which it is meant that they model the mixing of transport phenomena with diffusion. It is well known that in this kind of problem the behaviour of the solution is determined by the relative magnitudes of the convection and the diffusion terms. Diffusion-dominated problems behave like standard parabolic equations while advection-dominated ones, although theoretically parabolic, behave almost as if they were hyperbolic, except in some small regions, 'boundary layers', where diffusion effects reappear with startling consequences. In the case of the Navier–Stokes equations, the ratio of advection to diffusion is expressed by the Reynolds number. It is obtained by non-dimensionalizing the equations and has the form

$$Re = \frac{\rho U d}{\mu}. \tag{3.16}$$

It must be emphasized that a Reynolds number has no absolute meaning: it is a *relative* number. It enables problems in the *same geometry* with similar boundary conditions to be compared. In practice, high Reynolds number problems are difficult and must be handled with care but there is no absolute scale for 'large' or 'small' .

4. The Stokes problem: incompressibility and pressure

4.1. The continuous problem

We shall consider in this section the simplest possible incompressible flow problem, the steady-state Stokes problem, obtained from equations (3.1)–(3.2) by neglecting the time derivative and the inertial terms $\boldsymbol{u}\cdot\mathbf{grad}\,\boldsymbol{u}$. This *approximation* of the Navier–Stokes equations is valid for very low Reynolds numbers, that is for small velocities or high viscosity. The problem thus

becomes:

$$-2\mu\, \boldsymbol{A}\boldsymbol{u} + \mathbf{grad}\, p = \boldsymbol{f}, \tag{4.1}$$

$$\operatorname{div} \boldsymbol{u} = g, \tag{4.2}$$

$$\boldsymbol{u}|_\Gamma = 0. \tag{4.3}$$

In (4.2) we have introduced a non-zero right-hand side $g \in Q$. This is for the sake of generality and causes no additional difficulty. In most cases, we shall take $g = 0$. We shall describe how the Stokes problem can be seen as a constrained optimization problem and how pressure appears naturally as a Lagrange multiplier. This will enable us to apply the general results of Brezzi (1974), Babuška (1973) or Brezzi and Fortin (1991). This will also help us later in the construction of numerical algorithms for the computation of the pressure. First we define

$$a(\boldsymbol{u}, \boldsymbol{v}) = 2\mu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}x, \tag{4.4}$$

$$b(\boldsymbol{v}, q) = -\int_\Omega q \operatorname{div} \boldsymbol{v}\, \mathrm{d}x. \tag{4.5}$$

Clearly, problem (4.1)–(4.3) can be written in the form:

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}), & \forall \boldsymbol{v} \in V, \\ b(\boldsymbol{u}, q) = (g, q), & \forall q \in Q. \end{cases} \tag{4.6}$$

This problem is nothing but the optimality condition of a saddle-point problem,

$$\inf_{\boldsymbol{v} \in V} \sup_{q \in Q} \mu \int_\Omega |\boldsymbol{\varepsilon}(\boldsymbol{v})|^2\, \mathrm{d}x - \int_\Omega q \operatorname{div} \boldsymbol{v}\, \mathrm{d}x - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}x + \int_\Omega q\, q\, \mathrm{d}x, \tag{4.7}$$

which is equivalent to the contrained minimization problem,

$$\inf_{\operatorname{div} \boldsymbol{v} = g} \mu \int_\Omega |\boldsymbol{\varepsilon}(\boldsymbol{v})|^2\, \mathrm{d}x - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}x. \tag{4.8}$$

In this context, it is clear that the pressure may be seen as the Lagrange multiplier associated with the constraint $\operatorname{div} \boldsymbol{v} = g$. This will also remain true, in a generalized sense, for the full Navier–Stokes problem (3.1)–(3.3). If we now return to problem (4.6), we also see that we are now dealing with a *mixed variational formulation* (Brezzi 1974, Brezzi and Fortin 1991) and we have a general framework in which to study our problem. In general, the existence and uniqueness of the solution of a problem of type (4.6) requires two conditions. The first one is *coercivity* of the bilinear form $a(\cdot, \cdot)$ on V. In the case of the Stokes problem, in the setting defined above, this condition is immediately satisfied and is nothing but Korn's inequality, that is, there

exists a constant $\alpha > 0$ such that

$$\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}x \geq \alpha \|\boldsymbol{v}\|^2 \quad \forall \boldsymbol{v} \in V, \tag{4.9}$$

which holds for $V = (H_0^1(\Omega))^n (n = 2, 3)$ but also for more general boundary conditions (see Duvaut and Lions (1972)). The second condition is known as the inf–sup condition, which will be our terminology, but also as the Babuška–Brezzi condition or even the Ladyzhenskaya–Babuška–Brezzi (LBB) condition. It can be written as,

$$\inf_{\boldsymbol{v} \in V} \sup_{q \in Q} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{u}\|_V \|q\|_Q} \geq k_0 > 0. \tag{4.10}$$

This looks somewhat abstract and cumbersome. It means in fact that the operator B from V into Q', the dual of V, is surjective. In a more general form it can be written as

$$\inf_{\boldsymbol{v} \in V} \sup_{q \in Q} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{u}\|_V \|q\|_{Q/\ker B^t}} \geq k_0 > 0, \tag{4.11}$$

where

$$\ker B^t = \{q |\ b(\boldsymbol{v}, q) = 0, \forall \boldsymbol{v} \in V\} \tag{4.12}$$

and the quotient norm $\|q\|_{Q/\ker B^t}$ is defined by

$$\|q\|_{Q/\ker B^t} = \inf_{q_0 \in \ker B^t} \|q + q_0\|_Q. \tag{4.13}$$

Condition (4.11) then means that the operator B has a closed range in Q' and the p part of the solution is then only defined up to an element of $\ker B^t$. In our case, we have $Q = Q' = L^2(\Omega)$ and the operator B is the divergence operator from V into $L^2(\Omega)$. With $V = (H_0^1(\Omega))^2$, it is not surjective and $\ker B^t = \ker(\mathbf{grad})$ is the subspace of constants. Pressure will then be defined up to a constant. Whenever we have Neumann conditions on part of the boundary, we recover surjectivity and hence uniqueness.

4.2. The dual problem

It is usual, when a Lagrange multiplier is introduced to enforce a constraint, to consider the *dual problem*, that is the problem transformed into this new variable. It is obtained by changing the inf–sup problem (4.7) into a sup–inf problem through reversing the order of operations and eliminating $\boldsymbol{v}$ by performing the minimization in $\boldsymbol{v}$ for a given q. In our case, an easy calculation shows that the dual problem can be written as

$$\sup_q \frac{1}{2} \int_\Omega \boldsymbol{A}^{-1} \mathbf{grad}\, q \cdot \mathbf{grad}\, q\, \mathrm{d}x - \int_\Omega \boldsymbol{A}^{-1} \boldsymbol{f} \cdot \mathbf{grad}\, q\, \mathrm{d}x, \tag{4.14}$$

for which the optimality condition is

$$\operatorname{div} \boldsymbol{A}^{-1}\mathbf{grad}\, p = \operatorname{div} \boldsymbol{A}^{-1}\boldsymbol{f}. \tag{4.15}$$

As we shall see later, the properties of the discrete dual problem play a crucial role in the analysis of the numerical scheme.

4.3. The discrete problem

We are now in a position to consider discretizations of problem (4.6). To do so, we introduce finite-dimensional subspaces $V_h \subset V$ and $Q_h \subset Q$ and we consider the discrete analogue of (4.6),

$$\begin{cases} a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = (\boldsymbol{f}, \boldsymbol{v}_h), & \forall \boldsymbol{v}_h \in V_h, \\ b(\boldsymbol{u}_h, q_h) = (g, q_h), & \forall q_h \in Q_h, \end{cases} \tag{4.16}$$

where, as in (4.6) g will be zero in most cases. For such a *conforming approximation*, the general theory of Brezzi (1974) (see Brezzi and Fortin (1991)) applies directly. It relies on the discrete version of conditions (4.9) and (4.11). The first condition is trivial in the present case and follows directly from the inclusion $V_h \subset V$. To consider the second condition, we first identify Q_h and Q_h' just as we identified Q and Q', we let $B_h = \operatorname{div_h}$ be the discrete divergence operator from V_h into Q_h associated with the restriction of the bilinear form $b(\cdot,\cdot)$ to these spaces and let $B_h^t = \mathbf{grad}_h$ be its transpose,

$$\langle \operatorname{div_h} \boldsymbol{u}_h, q_h \rangle = b(\boldsymbol{u}_h, q_h) = \langle \boldsymbol{u}_h, \mathbf{grad}_h q_h \rangle, \ \boldsymbol{u}_h \in V_h, \ q_h \in Q_h. \tag{4.17}$$

In general, $\operatorname{div_h}$ is not the restriction of div to V_h. Indeed, from equation (4.17) we have

$$\operatorname{div_h} \boldsymbol{u}_h = P_{Q_h} \operatorname{div} \boldsymbol{u}_h, \tag{4.18}$$

where P_{Q_h} is the projection operator from Q onto Q_h. As we shall see later, in many actual cases $\operatorname{div_h} \boldsymbol{u}_h$ will be some average of $\operatorname{div} \boldsymbol{u}_h$. This also implies that the kernel of the discrete gradient $\mathbf{grad}_h$,

$$\ker \mathbf{grad}_h = \{ q_h \in Q_h \mid b(\boldsymbol{v}_h, q_h) = 0, \forall \boldsymbol{v}_h \in V_h \}, \tag{4.19}$$

is not necessarily the one-dimensional subspace of constants. Cases will arise in which nonconstant functions have a zero discrete gradient. Such cases will be pathological and will require special care if they are not simply avoided. We can now state the second condition as

$$\inf_{\boldsymbol{v}_h \in V_h} \sup_{q_h \in Q_h} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_V \|q_h\|_{Q_h/\ker \mathbf{grad}_h}} \geq k_h \geq k_0 > 0. \tag{4.20}$$

The first part ($\geq k_h$) is trivial in a finite dimensional setting. The really important requirement is the existence of a constant k_0 independent of h.

Given coercivity and the discrete inf–sup condition (4.20) we can apply the theory of Brezzi (1974) to obtain the existence and uniqueness of $(\boldsymbol{u}_h, p_h)$ in V_h and $Q_h/\ker \mathbf{grad}_h$ and we can state

Theorem 4.1 Let $(\boldsymbol{u}, p)$ be the solution of problem (4.6) and $(\boldsymbol{u}_h, p_h)$ be the solution of the discrete problem (4.16). We then have the error estimates:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V \leq C_1(1/\alpha, 1/k_h)\left\{\inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_V + \inf_{q_h \in Q_h} \|p - q_h\|_Q\right\}, \tag{4.21}$$

$$\|p - p_h\|_{Q/\ker(\mathbf{grad}_h)} \leq C_2(1/\alpha, 1/k_h^2)\left\{\inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_V + \inf_{q_h \in Q_h} \|p - q_h\|_Q\right\}. \tag{4.22}$$

□

It must be remarked that both constants C_1 and C_2 depend on $1/\alpha$ but that C_2 depends on $1/k_h^2$, which makes the approximation of pressure much more sensitive to a bad behaviour of k_h. In many cases where k_h is not bounded from below but depends on h, it is customary to see acceptable approximate velocities but a disastrous approximate pressure field. We shall develop later another approach to clarify this point. Before doing so, we shall present a criterion for the inf–sup condition and consider some classical examples.

4.4. The inf–sup condition and criteria

The question that now arises is to find some way of checking condition (4.20). Although this is not the only possibility, a quite convenient way is through a criterion introduced in Fortin (1977) which reduces the question to the construction of a suitable interpolation operator. The criterion can be found in a general setting in Brezzi and Fortin (1991). For the present purpose, we consider a special, albeit general enough case. As a starting point, we assume that the continuous inf–sup condition (4.11) holds, which is indeed the case for the problem considered. We then prove:

Lemma 4.1 Suppose that we can build an operator Π_h from V into V_h satisfying

$$b(\Pi_h \boldsymbol{v} - \boldsymbol{v}, q_h) = 0 \qquad \forall q_h \in Q_h, \tag{4.23}$$

$$\|\Pi_h \boldsymbol{v}\|_V \leq c\|\boldsymbol{v}\|_V, \tag{4.24}$$

with a constant c independent of h. Then the discrete inf–sup condition (4.20) holds

Proof. Indeed we have from (4.11), as $Q_h \subset Q$,

$$\sup_{\boldsymbol{v}\in V} \frac{b(\boldsymbol{v}, q_h)}{\|\boldsymbol{v}\|_V} \geq k_0 \|q_h\|_{Q/\ker B^t}. \tag{4.25}$$

But by (4.23) and (4.24), we may write

$$\begin{aligned} \sup_{\boldsymbol{v}_h\in V_h} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_V} &\geq \sup_{\boldsymbol{v}\in V} \frac{b(\Pi_h \boldsymbol{v}, q_h)}{\|\Pi_h \boldsymbol{v}\|_V} \\ &\geq \sup_{\boldsymbol{v}\in V} \frac{1}{c} \frac{b(\boldsymbol{v}, q_h)}{\|\boldsymbol{v}\|_V} \geq \frac{k_0}{c} \|q_h\|_{Q/\ker B^t}, \end{aligned} \tag{4.26}$$

hence the result. □

The use of Lemma 1 requires two things: finding a suitable class of elements, constructing Π_h which satisfies (4.23) and then checking that this operator is uniformly continuous in h, that is (4.24). This last requirement is generally purely technical although establising it could be quite intricate. It is also worth stating here an important fact about the operator Π_h.

Lemma 4.2 If the condition (4.23) holds, then

$$\ker B_h^t \subset \ker B^t, \tag{4.27}$$

that is there are no spurious zero-energy mode.

Proof. This is easily inferred from (4.23). Indeed we must show that any $q_h \in \ker B_h^t$, that satisfies

$$b(\boldsymbol{v}_h, q_h) = 0 \quad \forall \boldsymbol{v}_h \in V_h, \tag{4.28}$$

also satisfies

$$b(\boldsymbol{v}, q_h) = 0 \quad \forall \boldsymbol{v} \in V. \tag{4.29}$$

But $b(\boldsymbol{v}, q_h) = b(\Pi_h \boldsymbol{v}, q_h) = 0$ and the result is immediate. □

Let us come back to the problem of constructing the operator Π_h. This is often done in practice by starting from a standard interpolation operator and by correcting it by some local operations. The following lemma provides a general procedure to do so.

Lemma 4.3 Let us suppose that the finite element approximation has been chosen so that Proposition 2.2 or some analogous result applies with some suitable interpolation operator Π_1, satisfying the continuity requirement,

$$\|\Pi_1 \boldsymbol{v}\|_V \leq c_1 \|\boldsymbol{v}\|_V, \quad \forall \boldsymbol{v} \in V \tag{4.30}$$

We also suppose that there exists a second operator $\Pi_2 \in \mathcal{L}(V, V_h)$ satisfying

$$\|\Pi_2 (I - \Pi_1) \boldsymbol{v}\|_V \leq c_2 \|\boldsymbol{v}\|_V, \quad \forall \boldsymbol{v} \in V, \tag{4.31}$$

$$\int_\Omega \operatorname{div}(\boldsymbol{v} - \Pi_2 \boldsymbol{v})\, q_h \, dx = 0, \quad \forall \boldsymbol{v} \in V, \ \forall q_h \in Q_h, \tag{4.32}$$

where the constants c_1 and c_2 are independent of h. Then the operator Π_h defined by

$$\Pi_h \boldsymbol{u} = \Pi_1 \boldsymbol{u} + \Pi_2(\boldsymbol{u} - \Pi_1 \boldsymbol{u}), \ \boldsymbol{u} \in V, \tag{4.33}$$

satisfies (4.23) and (4.24).

Proof. It is easy to see that condition (4.23) holds. Indeed

$$\begin{aligned} b(\Pi_h w, q_h) &= b(\Pi_2(w - \Pi_1 w), q_h) + b(\Pi_1 w, q_h) \\ &= b(w - \Pi_1 w, q_h) + b(\Pi_1 w, q_h) \\ &= b(w, q_h). \end{aligned} \tag{4.34}$$

On the other hand,

$$\|\Pi_h w\|_v \le \|\Pi_2(w - \Pi_1 w)\|_V + \|\Pi_1 w\|_V \le (c_1 + c_2)\|w\|_V \tag{4.35}$$

so that condition (4.24) holds. □

In many cases, Π_1 will be the interpolation operator of Clement (1975) (cf. Proposition 2.2) in $H^1(\Omega)$ for which we have

$$\sum_K h_K^{2r-2} |\boldsymbol{v} - \Pi_1 \boldsymbol{v}|_{r,K}^2 \le c \, \|\boldsymbol{v}\|_{1,\Omega}^2, \quad r = 0, 1 . \tag{4.36}$$

Taking $r = 1$ in (4.36) and using the triangle inequality

$$\|\Pi_1 \boldsymbol{v}\|_V \le \|\boldsymbol{v} - \Pi_1 \boldsymbol{v}\|_V + \|\boldsymbol{v}\|_V \tag{4.37}$$

yields (4.30).

4.5. The matrix form of the discrete problem

Suppose that we are given a basis $\{\phi_i^V\}_{1 \le i \le N}$ of V and a basis $\{\psi_k^Q\}_{1 \le k \le M}$ of Q. We can define the matrices

$$\mathcal{A}_{i,j} = a(\phi_i^V, \phi_j^V), \tag{4.38}$$

$$\mathcal{B}_{i,k} = b(\phi_i^V, \psi_k^Q). \tag{4.39}$$

Matrix $\mathcal{A}$ is positive definite while $\mathcal{B}$ is a rectangular matrix. We shall also need later the mass matrices

$$\mathcal{M}_{i,j}^V = (\phi_i^V, \phi_j^V)_{0,\Omega}, \tag{4.40}$$

$$\mathcal{M}_{k,l}^Q = (\psi_k^Q, \psi_l^Q)_{0,\Omega}. \tag{4.41}$$

We now consider the discrete problem (4.16) and we write

$$\begin{cases} \boldsymbol{u}_h = \sum_i \boldsymbol{U}_i \phi_i^V, \\ p_h = \sum_k \boldsymbol{P}_k \psi_k^Q. \end{cases} \tag{4.42}$$

Using this notation, the discrete problem may be written as

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^t & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{U} \\ \boldsymbol{P} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{G} \end{pmatrix}. \tag{4.43}$$

We therefore have to solve a symmetric indefinite system. It is possible, as $\mathcal{A}$ is invertible, to eliminate $\boldsymbol{U}$ from this problem to get a problem in $\boldsymbol{P}$ only:

$$\mathcal{B}\mathcal{A}^{-1}\mathcal{B}^t\boldsymbol{P} = \boldsymbol{G} - \mathcal{B}(\mathcal{A})^{-1}\boldsymbol{F}. \tag{4.44}$$

We shall come back later to numerical methods adapted to this problem.

4.6. Eigenproblems associated with the discrete inf–sup condition

It should be clear from the earlier analysis that the discrete inf–sup condition is closely related to the behaviour of the dual problem, in particular the discrete dual problem (4.44). Let us indeed go back to (4.20) and let us rewrite it in the notation of the previous subsection. We get

$$\inf_{\boldsymbol{V}\in V_h} \sup_{\boldsymbol{Q}\in Q_h} \frac{\langle \mathcal{B}\boldsymbol{V}, \boldsymbol{Q}\rangle}{\langle \mathcal{A}\boldsymbol{V}, \boldsymbol{V}\rangle^{1/2}\langle \mathcal{M}^Q\boldsymbol{Q}, \boldsymbol{Q}\rangle^{1/2}} = k_h, \tag{4.45}$$

where we have made the assumption that $\langle \mathcal{A}\boldsymbol{V}, \boldsymbol{V}\rangle^{1/2}$ is employed as a norm on V_h. This involves a Rayleigh's quotient for the *singular value decomposition* of the matrix $\mathcal{B}$ with the norms defined by $\mathcal{A}$ and $\mathcal{M}^Q$ on V_h and Q_h respectively. This can be reduced to solving the generalized eigenvalue problem,

$$\mathcal{B}\mathcal{A}^{-1}\mathcal{B}^t\boldsymbol{Q}_i = \mu_i^2\mathcal{M}^Q\boldsymbol{Q}_i. \tag{4.46}$$

The square root of the smallest eigenvalue is nothing but the constant k_h of (4.20) while the square root of the largest one is the norm $\|b\|$ of the bilinear form $b(\cdot,\cdot)$. For more details, we refer to Brezzi and Fortin (1991). This argument shows that all kinds of behaviour is possible: the correct case is when the eigenvalues are bounded away from zero. When some eigenvalues vanish with h, part of the solution will be spoiled. We refer to Malkus (1981) where these eigenvalues have been computed numerically for some elements. A more complete discussion of similar eigenvalue problems and of the condition number of associated systems can be found in Fortin and Pierre (1992).

5. Finite elements for incompressible problems

In this section, we shall present, in a general framework, some classical examples of finite element approximations to the equations of incompressible materials. The problem is *a priori* simple. We are looking for a velocity field in $H_0^1(\Omega))^2$, which implies that all classical constructions hold. In the

same way, the pressure which is sought only in $L^2(\Omega)$ can be approximated by a very wide choice of elements. Standard conforming elements (built for $H^1(\Omega)$) will evidently be suitable and we shall say in that case that we have a *continuous pressure approximation.* On the other hand, we are allowed to avoid any continuity of the discrete pressure at interfaces and to use *discontinuous pressure approximations.* As we shall see in the examples, this last case will ensure a better conservation of mass. The difficulty which arises is that our approximations of velocity and pressure cannot be chosen independently but must satisfy a compatibility condition: the discrete inf–sup condition (4.20). Our goal will therefore be to build approximations satisfying this condition while preserving simplicity and efficiency.

5.1. Exact incompressibility

A natural idea when one comes to the problem of approximating divergence-free problems is to try to enforce the constraint strongly, that is, at every point. This can be done quite easily. Indeed, given a choice of a space V_h for the approximate velocities, it would be sufficient to take Q_h so that it contains $\operatorname{div} V_h$ to ensure that the divergence of the solution is zero everywhere. Just as many simple ideas, this one leads to a dead-end. What happens, at least for low-degree elements, is that the solution is overconstrained and we have a *locking phenomenon*, that is the only function satisfying the divergence-free constraint is the function identically zero. This is the case in the following simple example.

Example 5.1 (The P_1–P_0 approximation.) We approximate velocity by the simplest finite element: piecewise linear functions on triangles. The divergence is then a subspace of the space of piecewise constants so that using this space for Q_h enforces the divergence-free condition exactly. A simple count, using Euler's relations on a triangulation, however, shows that, on a general mesh, the number of constraints is larger than the number of degrees of freedom and that we have locking. It must, however, be noted that on a composite mesh where triangles are obtained by dividing quadrilaterals by their diagonals (Figure 5.1), a linear dependence appears between the constraints so that non-trivial solutions exist. Nevertheless, the resulting approximation does not satisfy the discrete inf–sup condition. □

Example 5.2 (Quadrilateral elements.) The reader may easily check that the same locking phenomenon will appear on rectangular elements (on a regular grid for instance) if one tries to impose an *exact* divergence-free condition to a bilinear or biquadratic approximation of velocities. □

Example 5.3 (Second-order triangular elements.) The counting procedure also shows that, on a general triangular mesh, piecewise quadratic

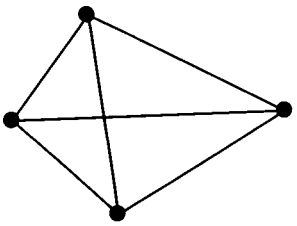

Fig. 5.1. A quadrilateral subdivided by diagonals.

divergence-free elements exist but require far too many triangles to be efficient. However, using the same mesh as in Figure 5.1, it is possible to get a divergence-free approximation satisfying the discrete inf–sup condition; but it is then necessary to filter a spurious mode from the pressure approximation. We refer to Brezzi and Fortin (1991) for details. □

Remark 5.1 The previous example is directly related to the composite approximation of Fraeijs de Veubeke and Sander (1968) for plate problems (see also Ciavaldini and Nédélec (1974)). Those problems indeed require C^1-continuity, and a composite element of degree three can be built on the mesh of Figure 5.1. Taking the curl of this approximation yields a divergence-free function, which is piecewise quadratic and C^0-continuous. The construction is therefore based on the approximation of the *stream function* in $H_0^2(\Omega)$. It must be noted that no similar composite constructions are known for three-dimensional problems. □

Remark 5.2 (Higher order methods.) We would like to recall briefly the statement of a basic result by Scott and Vogelius (1985) which, roughly speaking, says: under minor assumptions on the decomposition $\mathcal{T}_h$ (in triangles) the pair $V_h = (\mathcal{L}_k^1)^2$, $Q_h = \mathcal{L}_{k-1}^1$ satisfies the inf–sup condition for $k \geq 4$. This, in a sense, settles the matter as far as higher order methods are concerned, and leaves only the problem of finding stable lower order approximations. It must, however, be noted that some instabilities might in certain cases remain in the pressure, although they could be filtered out *a posteriori*. In fact the restrictions to which we alluded earlier are that the sides of the triangles should not be collinear as in the special grid of Figure 5.1, which reduces the number of linearly dependent constraints, leaving some of the pressure degrees of freedom unused. This being said, the use of high-order methods is not very popular as they require the delicate manipulation of high-degree polynomials. It has however gained a new popularity.

We have seen that exactly divergence-free methods are delicate, requiring high-order elements or special grids. This leads us to try enforcing the divergence-free condition only approximately, in the hope of obtaining simpler constructions. We have already noted (cf. equation (4.17)) that we have

to deal with a discrete divergence operator div_h which is the projection on Q_h of the divergence operator. If Q_h is smaller, or V_h larger, this projection will effectively weaken the divergence-free condition. The effect will be, on one hand, that the verification of the discrete inf–sup condition (4.20) will be easier. On the other hand, the accuracy of the approximation would evidently be impaired by taking Q_h too small so that we shall privelege the enrichment of V_h as a potential cure to our difficulties.

5.2. Simple constructions for approximately divergence-free elements

We shall now introduce a simple and general way of satisfying the inf–sup condition. The basic idea is indeed very simple: the discrete inf–sup condition involves a supremum over $v_h \in V_h$. Making the space V_h larger will make the supremum grow and will intuitively make the condition easier to fulfil. This technique can be further extended to composite elements but for the sake of simplicity, it is worth considering the simpler case first.

The idea of *enriched* elements has been used several times, starting with Crouzeix and Raviart (1973) for discontinuous pressures and Arnold, Brezzi and Douglas (1984) and Arnold, Brezzi and Fortin (1984) for continuous pressures. We present it in the general form given by Brezzi and Pitkäranta (1984) (see also Stenberg (1984)). It consists essentially in stabilizing an element by an enrichment of the velocity field by **bubble functions**, that is functions having their support restricted to one element and vanishing on the boundary of this element. The simplest bubble function is the conforming bubble function, denoted $b_{3,K}$. It is a polynomial function of degree three. If we denote by $\lambda_1, \lambda_2, \lambda_3$, the barycentric coordinates of the triangle we then have $b_{3,K} = \lambda_1\lambda_2\lambda_3$. We associate with the finite element discretization $Q_h \subset L^2(\Omega)$ the space

$$M(\mathbf{grad}\, Q_h) = \{\beta \mid \beta|_K = b_{3,K}\mathbf{grad}\, q_h|_K \quad \text{for some } q_h \in Q_h\}. \tag{5.1}$$

In other words, the restriction of a function $\beta \in M(\mathbf{grad}\, Q_h)$ to an element K is the product of the P_3-bubble functions $b_{3,K}$ and the gradient of a function from $Q_h|_K$.

Remark 5.3 Notice that the space $M(\mathbf{grad}\, Q_h)$ is not defined through some basic space $\widehat{M}$ on the reference element. This can be easily done, if one wants to, in the case of *affine* elements, for all the reasonable choices of Q_h. However this is clearly unnecessary: if we know how to compute q_h on K we also know how to compute $\mathbf{grad}\, q_h$ and there is no need for a reference element. □

We now turn to prove two results, concerning continuous or discontinuous pressures.

Proposition 5.1 (Stability of continuous pressure elements.) We suppose

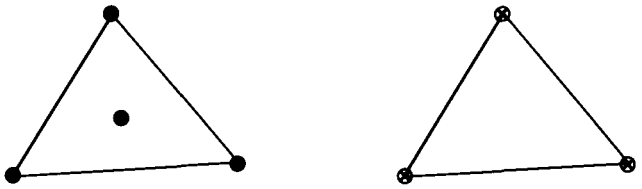

Fig. 5.2. The MINI element.

that there exists $\Pi_1 \in \mathcal{L}(V, V_h)$ satisfying (4.30), that we have $Q_h \subset H^1(\Omega)$ and that $M(\mathbf{grad}\, Q_h)$ is defined as in (5.1). Then the pair (V_h, Q_h) is a stable element, in the sense that it satisfies the inf–sup condition.

Proof. We shall use Lemma 4.3. We already have our operator Π_1 by assumption. We only need to construct Π_2. We define $\Pi_2 : V \to M(\mathbf{grad}\, Q_h)$, on each element, by requiring

$$\begin{cases} \Pi_2 \boldsymbol{v}|_K \in M(\mathbf{grad}\, Q_h)|_K = b_{3,K} \mathbf{grad}\, Q_h|_K, \\ \displaystyle\int_K (\Pi_2 \boldsymbol{v} - \boldsymbol{v}) \cdot \mathbf{grad}\, q_h \; \mathrm{d}x = 0, \quad \forall q_h \in Q_h|_K. \end{cases} \tag{5.2}$$

Problem (5.2) has obviously a unique solution. It is clear that Π_2 satisfies (4.31) of Lemma 4.3. Finally (4.30) follows by a scaling argument. We thus have the desired result. □

Corollary 5.1 Assume that $Q_h \subset Q$ is any space of continuous piecewise smooth functions. If $(\mathcal{L}_1^1)^2 \oplus M(\mathbf{grad}\, Q_h) \subset V_h$ then the pair (V_h, Q_h) satisfies the inf–sup condition.

Proof. Continuity and piecewise smoothness imply that $Q_h \subset H^1(\Omega)$. The condition $(\mathcal{L}_1^1)^2 \subset V_h$ implies the existence of Π_1 satisfying condition (4.30), and condition $M(\mathbf{grad}\, Q_h) \subset V_h$ is by hypothesis. Hence we can apply Proposition 5.1. □

These results apply, for instance, to the enriched Taylor–Hood element and to the families introduced in Arnold, Brezzi and Fortin (1984).

Example 5.4 (The MINI element.) The first family is defined by

$$V_h = (\mathcal{L}_k^1 \oplus B_{k+2})^2, \quad Q_h = \mathcal{L}_k^1, k \geq 1, \tag{5.3}$$

where B_{k+2} is defined as in (2.14). The simplest of these elements is the so-called MINI element. It is obtained by taking $k = 1$ in (5.3). This means that a cubic bubble, $(k+2 = 3)$, is added to a simple piecewise linear approximation of velocity while pressure remains piecewise linear. This element is sketched in Figure 5.2. The corresponding **equal interpolation** element, using piecewise linear approximations for both velocity and pressure is not stable in the sense that it does not satisfy the inf–sup condition. This is, in fact, the case for all equal interpolation approximations. □

Fig. 5.3. Enriched Taylor–Hood element.

Example 5.5 The second family is defined by

$$V_h = (\mathcal{L}_k^1 \oplus B_{k+1})^2, \quad Q_h = \mathcal{L}_{k-1}^1, \; k \geq 2. \tag{5.4}$$

In the simplest case, ($k = 2$), a cubic bubble is added to a piecewise quadratic approximation of velocity while pressure is approximated by piecewise linear functions as in the previous example. This element is sketched in Figure 5.3 Without bubbles, this element is known as the *Taylor–Hood element* and it is already stable. The proof of this will be considered later, as it requires a special technique. □

We turn now to the case of discontinuous pressure elements. Many reasons may lead us to consider such approximations. Probably the most important one is probably the better approximation to the equation of conservation of mass generated by such elements, in comparison with dicontinuous pressure elements. In fact, whenever Q_h contains piecewise constant functions, the divergence-free condition contains, as a particular case, the condition

$$\int_K \operatorname{div} \boldsymbol{v}_h \, \mathrm{d}x = 0, \quad \forall K \in \mathcal{T}_h, \tag{5.5}$$

which means that the average divergence is null on every element or, equivalently, that mass is conserved on every element. In the case of continuous pressure approximations, the divergence-free condition is also averaged, but the averages cannot be reduced to a local conservation property. In hard cases this may have important consequences (Fortin and Pelletier (1989)). Discontinuous pressures are also important because they can be combined with a penalty method to eliminate pressure as an unknown, as we shall see in Section 8. Before stating the general result, we shall consider a simple special case which will be the basis for the general setting.

Example 5.6 (The P_2–P_0 and Q_2–Q_0 elements.) These are the basic and simplest stable discontinuous pressure elements. We shall only consider the triangular case in detail as the quadrilateral case can be treated in essentially the same way. The element by itself has no particular property except that pressure is approximated with very low precision. If we refer to estimates (4.22) and (4.23), this implies that we do not achieve the full accuracy expected from second-degree polynomials employed to approximate velocity.

Fig. 5.4. The P_2–P_0 element.

The accuracy of the element is thus not optimal and it is not recommnded in practice.

To check the inf–sup condition, we shall use Lemma 4.1 and build an operator Π_h satisfying the conditions (4.23) and (4.24). To do so we start from the standard interpolation operator Π_1 of Proposition 2.2 and we modify the midside values so that on every side S of K, the resulting new interpolant $\tilde{\Pi}_1 \boldsymbol{v}$ satisfies

$$\int_S (\tilde{\Pi}_1 v_i - v_i)\,\mathrm{d}s = 0 \quad \forall \boldsymbol{v} \in V, i = 1,2, \tag{5.6}$$

where the v_i are the components of $\boldsymbol{v}$. Formally, we can work again with Lemma 4.3 and define, on every element K, $\Pi_{2|_K}$ by

$$\begin{cases} \Pi_2 \boldsymbol{v} \in P_2(K), \\ \Pi_2 \boldsymbol{v}_{|_K}(M) = 0, \quad \text{for any vertex } M \text{ of } K, \\ \displaystyle\int_S \Pi_2 \boldsymbol{v}\ \mathrm{d}s = \int_S \boldsymbol{v}\,\mathrm{d}s, \quad \text{for every side } S \text{ of } K. \end{cases} \tag{5.7}$$

One then defines $\tilde{\Pi}_1$ by

$$\tilde{\Pi}_1 \boldsymbol{v} = \Pi_1 \boldsymbol{v} + \Pi_2(\boldsymbol{v} - \Pi_1 \boldsymbol{v}). \tag{5.8}$$

It is clear that that $\tilde{\Pi}_1$ satisfies condition (4.23) for we have

$$\int_K \operatorname{div}(\Pi_2 \boldsymbol{v} - \boldsymbol{v})\,\mathrm{d}x = \int_{\partial K} (\Pi_2 \boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{n}\,\mathrm{d}s = 0. \tag{5.9}$$

As to the continuity property, it follows by a scaling argument as

$$\begin{aligned} |\Pi_2 \boldsymbol{v}|_{1,K} = |\widehat{\Pi_2 \boldsymbol{v}}|_{1,\hat{K}} &\le c(2,\theta_0)\|\hat{\boldsymbol{v}}\|_{1,\hat{K}} \\ &\le c(2,\theta_0)(h_K^{-1}|\boldsymbol{v}|_{0,K} + |\boldsymbol{v}|_{1,K}), \end{aligned} \tag{5.10}$$

where $c(2,\theta_0)$ is a constant, depending on the degree of the polynomials employed, which is 2, and on the minimum angle of the mesh as in (2.17). Using this result and the properties of Π_1 yields the result. □

Proposition 5.2 (Stability of discontinuous pressure elements.) Let us

suppose that there exists $\tilde{\Pi}_1 \in \mathcal{L}(V, V_h)$ satisfying

$$\int_K \operatorname{div}\,(\boldsymbol{v} - \tilde{\Pi}_1 \boldsymbol{v})\,\mathrm{d}x = 0 \quad \forall K \in \mathcal{T}_h \tag{5.11}$$

and that $M(\mathbf{grad}\, Q_h) \subset V_h$ as defined in (5.1). Then the pair (V_h, Q_h) is a stable element, in the sense that it satisfies the inf–sup condition.

Proof. We shall proceed by applying Lemma 4.3. We take $\tilde{\Pi}_1$ satisfying (5.11) as operator Π_1. We are not going to define Π_2 on all of V, but only in the subspace

$$V^0 = \left\{ \boldsymbol{v} \mid \boldsymbol{v} \in V,\ \int_K \operatorname{div} \boldsymbol{v}\,\mathrm{d}x = 0, \quad \forall K \in \mathcal{T}_h \right\} \tag{5.12}$$

For every $\boldsymbol{v} \in V^0$ we construct $\Pi_2 \boldsymbol{v} \in M(\mathbf{grad}\, Q_h)$ by requiring that, on each element K,

$$\begin{cases} \Pi_2 \boldsymbol{v}|_K \in M(\mathbf{grad}\, Q_h)|_K = b_{3,K} \mathbf{grad}\, Q_h|_K, \\ \displaystyle\int_K \operatorname{div}\,(\Pi_2 \boldsymbol{v} - \boldsymbol{v})\, q_h\,\mathrm{d}x = 0, \quad \forall q_h \in Q_h|_K. \end{cases} \tag{5.13}$$

Note that (5.13) is uniquely solvable if $\boldsymbol{v} \in V^0$ since the divergence of a bubble function always has zero mean value (hence the number of nontrivial equations is equal to $\dim(Q_h|_K) - 1$, which is equal to the number of unknowns; the nonsingularity then follows easily). It is clear that Π_2, as given by (5.13), will satisfy (4.31) for all $\boldsymbol{v} \in V^0$. We have to check that

$$\|\Pi_2 \boldsymbol{v}\|_1 \leq c\, \|\boldsymbol{v}\|_1, \tag{5.14}$$

which actually follows again by a scaling argument. It is then easy to see that the operator

$$\Pi_h = \tilde{\Pi}_1 + \Pi_2 (I - \tilde{\Pi}_1) \tag{5.15}$$

satisfies the condition of Lemma 4.3 and the inf-sup condition follows. □

Corollary 5.2 (Bi-dimensional triangular case.) Let us assume that $Q_h \subset Q$ is any space of piecewise smooth functions and suppose that

$$(\mathcal{L}_2^1)^2 \oplus M(\mathbf{grad}\, Q_h) \subset V_h.$$

Then the pair (V_h, Q_h) satisfies the inf-sup condition.

Proof. The condition $(\mathcal{L}_2^1)^2 \subset V_h$ implies that we can construct $\tilde{\Pi}_1$ as in Example 5.6. On the other hand we have $M(\mathbf{grad}\, Q_h) \subset V_h$, so that we can apply the previous Proposition 5.2. □

Propositions 4.1, 4.2 and 4.3 require a few comments. They show that almost any element can be stabilized by using bubble functions. For continuous pressure elements this procedure is mainly useful in the case of triangular elements. For discontinuous pressure elements it is possible to stabilize

Fig. 5.5. The Crouzeix–Raviart element.

Fig. 5.6. The Q_2–P_1 element.

elements provided that they are already stable for piecewise constant pressure field. Examples of such a procedure can be found in Fortin and Fortin (1985a). Stability with respect to piecewise constant pressure implies that at least one degree of freedom on each side or face of the element is linked to the normal component of the velocity (Bernardi and Raugel (1981) or Fortin (1981)). Let us now consider a few examples of discontinuous pressure elements.

Example 5.7 (The Crouzeix and Raviart element.) We take Q_h to be the space of piecewise linear discontinuous functions. The previous construction then consists in adding cubic bubbles to a piecewise quadratic approximation of the velocity. This element is sketched in Figure 5.5. It provides second-order accuracy and is probably one of the best choices among stable triangular elements. □

This element has a rectangular (or even isoparametric) counterpart which is worth presenting. One interesting fact is that the triangular and rectangular versions are compatible and can be used inside a mixed mesh.

Example 5.8 (The Q_2–P_1 element and generalizations.) Let us consider an approximation of the velocity by a full biquadratic approximation and of the pressure by piecewise *linear* discontinuous functions. It can then be checked that the element is stable using the same kind of argument as in Corollary 5.2. This element, sketched in Figure 5.6, is one of the most popular elements for the approximation of incompressible flows. Although the previous results, as stated, can only be applied to the triangular case, the rectangular case and its isoparametric counterpart can be handled along the same lines. The idea is that once the constant part of the pressure is controlled by integrals on the boundary of the element, one may use *internal*

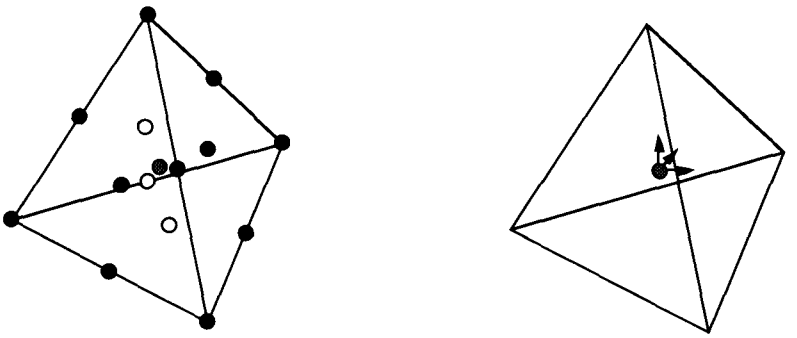

Fig. 5.7. Three-dimensional Crouzeix–Raviart element.

nodes to control the remaining part. In the case of a full biquadratic Q_2 approximation of the velocity, we have two internal degrees of freedom so that a P_1 pressure field can be used, but not a bilinear Q_1, as this would require three internal nodes. Evidently, one could enrich the approximation of the velocity to accommodate any degree of approximation of the pressure (Fortin and Fortin 1985a). It can be easily checked that for $k \geq 3$ a Q_k approximation of the velocity can be combined with a P_{k-1} or a Q_{k-1} approximation of the pressure. The case of $k = 1$ is pathological and will be discussed later. □

Example 5.9 (Three-dimensional tetrahedral discontinuous pressure elements.) The same arguments can be directly translated to the three-dimensional case (cf. Fortin (1981) or Stenberg (1987)). The main difference is that, in order to control the piecewise constant part of the pressure, one needs to use degrees of freedom on the faces of tetrahedra rather than on the edges. The equivalent of the operator $\tilde{\Pi}_1$ requires the integration of fluxes on faces; this requires the use of polynomials of degree greater than or equal to three if we only want to enrich by internal nodes. However, there exists a three-dimensional Crouzeix and Raviart element as sketched in Figure 5.7. It is obtained by enriching a second-degree element (with ten degrees of freedom on vertices and on the edges) by one cubic bubble on each face plus one fourth-degree internal bubble. Moving to higher degree polynomials, one may similarly build enriched elements with any order of accuracy. Finally with polynomials of degree higher than or equal to nine, one may build exactly divergence-free elements, as in the result of Scott and Vogelius (1985) discussed earlier. □

Example 5.10 (Three-dimensional hexahedral discontinuous pressure elements.) It can easily be checked that the three-dimensional Q_2–P_1 element sketched in Figure 5.8 is also a stable element. The Q_2–Q_1 element is not stable. For $k \geq 3$, a Q_k–P_{k-1} approximation is stable and for $k \geq 4$, so will Q_k–Q_{k-1}. □

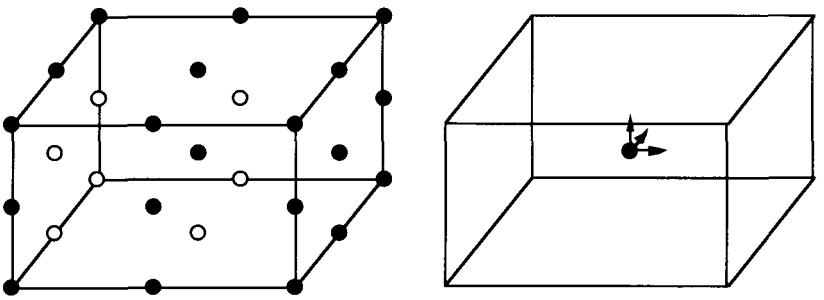

Fig. 5.8. Three-dimensional Q_2–P_1 element.

5.3. Nonconforming elements

We have just seen that it is possible to build approximately divergence-free approximations by enriching the approximation of the velocity. A different way to obtain stable approximations is to employ nonconforming elements, that is elements for which continuity requirements at interfaces have been relaxed. Using nonconforming elements implies that the variational formulation must be modified. In the Stokes problem, for instance, one must define discrete versions of the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$,

$$a_h(\boldsymbol{u},\boldsymbol{v}) = 2\mu \sum_K \int_K \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x, \tag{5.16}$$

$$b_h(\boldsymbol{v},q) = \sum_k \int_K q\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x. \tag{5.17}$$

These discrete forms are defined even for functions which are discontinuous at interfaces, as only derivatives inside elements are involved. The discrete problem can now be written

$$\begin{cases} a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) + b_h(\boldsymbol{v}_h,p_h) = (\boldsymbol{f},\boldsymbol{v}_h), & \forall \boldsymbol{v}_h \in V_h, \\ b_h(\boldsymbol{u}_h,q_h) = 0, & \forall q_h \in Q_h. \end{cases} \tag{5.18}$$

It is then possible to perform an error analysis of the problem. We refer to Brezzi and Fortin (1991) or to the original work of Crouzeix and Raviart (1973) for precise results the development of which is beyond the scope of this article. Let us simply say that nonconformity introduces additional consistency terms in the error analysis. These terms have to be properly bounded and the key for this is the *generalized patch-test*: 'for a nonconforming approximation of degree k to be optimal with respect to error estimates, the moments $\int_S \boldsymbol{v}_h p_{k-1}\,\mathrm{d}s$, must be continuous at any interface S, for any polynomial p_{k-1} of degree $k-1$' . The simplest of these elements is described in the following example. It was introduced in Crouzeix and Raviart (1973).

Fig. 5.9. The nonconforming P_1 element.

Example 5.11 (The nonconforming P_1 element.) We consider an approximation of the velocity by functions which are piecewise linear but are continuous only at midside points at element interfaces. This implies that $\int_S \boldsymbol{v}\,\mathrm{d}s$ is continuous as the midpoint rule is exact for polynomials of degree one and the patch–test is therefore satisfied. The pressure is piecewise constant and the element is sketched in Figure 5.9. This is the simplest first-order accurate element for incompressible problems. As pressure is discontinuous, one has local conservation of mass. The three-dimensional analogue is readily built, using values at the barycentre of the faces as degrees of freedom. □

It is also possible to construct higher order nonconforming elements. This is easily done for odd degree polynomials. One can find, for instance, a third-order nonconforming element in the paper of Crouzeix and Raviart (1973) in which a polynomial of degree three, enriched by bubbles of degree four, is employed. Continuity is then required at three Gauss–Legendre points on each element side. This implies that the element passes the correct patch test and the values at those Gauss–Legendre points can be used, with the addition of some internal nodes, as degrees of freedom. For even degree polynomials, a pathology arises and a different way must be found, as described in the next example.

Example 5.12 (The Fortin–Soulié nonconforming element.) It is easy to see that, in the two-dimensional case, the construction of a nonconforming element of degree two (or more generally of even degree), leads to unexpected difficulties. To satisfy the patch test and obtain the correct accuracy, one should ensure continuity at the two Gauss–Legendre points on the sides of elements. The trouble is that these six points cannot be used as degrees of freedom for a polynomial of degree two as one would like to do following the previous example: there exists a *nonconforming bubble* which vanishes at all six Gauss–Legendre points. It is expressed, in barycentric coordinates, as

$$b_{nc}(\lambda_1, \lambda_2, \lambda_3) = 2 - 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2). \tag{5.19}$$

The way around this difficulty is to construct second-order nonconforming methods in the same way as one built the element of Example 5.5: by enriching a standard conforming element of degree two by the nonconforming bubble (5.19). We refer to Fortin and Soulié (1983) for details. The degrees

of freedom are the same as in a Crouzeix–Raviart element and only the bubble function has to be modified in the code, essentially a one line change. The advantage is that now only polynomials of degree two have to be manipulated. A three-dimensional version has also been derived in Fortin (1985). □

Finally, let us note that it is possible to build quadrilateral nonconforming elements along the same lines, that is by enriching a standard element by a function satisfying the patch test. A Q_1 nonconforming element can, for example, be obtained by adding to the standard conforming element a function of the form $\phi(x, y) = xy$ on $]-1, 1[\times]-1, 1[$ (Fortin and Soulié, 1983). It is also possible to add a function of the form $x^2 - y - y^2$ to a P_1 approximation (Rannacher and Turěk, 1992).

5.4. Taylor–Hood elements and generalizations

There exists another class of stable elements which is not covered by the previous analysis and which are worth a presentation. This class contains the Taylor–Hood element and its generalizations (Hood and Taylor, 1973; Bercovier and Pironneau, 1977; Brezzi and Falk, 1991). They essentially consist of taking, for triangular elements

$$V_h = \mathcal{L}_k^1, \quad Q_h = \mathcal{L}_{k-1}^1, \tag{5.20}$$

that is continuous pressure elements with the pressure one degree lower than the velocity. This yields the right order of accuracy as one only approximates pressure in $L^2(\Omega)$. The corresponding quadrilateral elements are also widely employed and the three-dimensional counterpart is quite popular. Because it contains an important idea, Verfürth's trick, we rapidly sketch the proof of stability for the original Taylor–Hood elemencorresponding to $k = 2$ in (5.20). The proof proceeds in two steps, the first being very general.

Lemma 5.1 Let Ω be a bounded domain of $\mathbf{R}^n$ with Lipschitz continuous boundary. Let $V_h \subset (H_0^1(\Omega))^2 = V$ and $Q_h \subset H^1(\Omega)$. Suppose that there exists a linear operator Π_h^0 from V into V_h and a constant c, independent of h, such that

$$\|\boldsymbol{v}_h - \boldsymbol{v}\|_{r,\Omega} \leq c \sum_K \left(h_K^{2-2r} \|\boldsymbol{v}\|_{1,K}^2 \right)^{1/2}, \quad \forall \boldsymbol{v} \in V, r = 0, 1. \tag{5.21}$$

Then there exist two positive constants c_1 and c_2 such that for every $q_h \in Q_h$

$$\sup_{\boldsymbol{v}_h \in V_h} \frac{\int_\Omega q_h \operatorname{div} \boldsymbol{v}_h \, \mathrm{d}x}{\|\boldsymbol{v}_h\|_1} \geq c_1 \|q_h\|_{0/\mathrm{R}} - c_2 \sum_k \left(h_k^2 \|\mathbf{grad}\, q_h\|_{0,K}^2 \right)^{1/2}. \tag{5.22}$$

We refer to Brezzi and Fortin(1991) for the proof. Let us remark again that this is general and holds for any continuous pressure approximation. Now

let us return to the special case of the quadratic–linear approximation of Taylor and Hood.

Lemma 5.2 (Stability of the Taylor–Hood element.) Let $V_h = (\mathcal{L}_2^1)^2 \cap (H_0^1(\Omega))^2$ and $Q_h = \mathcal{L}_1^1$. Then, if any element of $\mathcal{T}_h$ has no more than one edge on the boundary, there exists a positive constant c_3 such that for every $q_h \in Q_h$,

$$\sup_{\boldsymbol{v}_h \in V_h} \frac{\int_\Omega q_h \operatorname{div} \boldsymbol{v}_h \, \mathrm{d}x}{\|\boldsymbol{v}_h\|_1} \geq \left(\sum_K h_K^2 |q_h|_{1,K}^2 \right)^{1/2}. \tag{5.23}$$

Proof. We shall prove the result by constructing a suitable $\bar{\boldsymbol{v}}_h$. Let $q_h \in Q_h$ be given and let K be an element of $\mathcal{T}_h$. We define $\bar{\boldsymbol{v}}_h$ on K by

$$\begin{cases} \bar{\boldsymbol{v}}_h = 0 \ \text{ at the vertices of } K, \\ \bar{\boldsymbol{v}}_h = -\boldsymbol{t}_e (\mathbf{grad}\, q_h \cdot \boldsymbol{t}_e) |e|^2, \end{cases} \tag{5.24}$$

at the midpoint of every edge e of K, denoting by $|e|$ the length of e and by $\boldsymbol{t}_e$ the unit tangent vector to e, with some chosen orientation. One easily checks that

$$\|\bar{\boldsymbol{v}}_h\|_{1,K} \leq c h_K |q_h|_{1,K}. \tag{5.25}$$

Now, we use a quadrature formula, which is exact for any polynomial of degree 2,

$$\int_K p_2(x) \, \mathrm{d}x = \frac{\operatorname{meas}(K)}{3} \sum_M p_2(M), \tag{5.26}$$

where the sum is taken over the midpoints M of the edges of K. We then have, with the choice (5.24),

$$\begin{aligned} \int_\Omega q_h \operatorname{div} \bar{\boldsymbol{v}}_h \, \mathrm{d}x &= -\int_\Omega \mathbf{grad}\, q_h \cdot \bar{\boldsymbol{v}}_h \, \mathrm{d}x \\ &= -\textstyle\sum_K \int_K \mathbf{grad}\, q_h \cdot \bar{\boldsymbol{v}}_h \, \mathrm{d}x \\ &= -\textstyle\sum_K \sum_M (\mathbf{grad}\, q_h \cdot \bar{\boldsymbol{v}}_h)(M) \dfrac{\operatorname{meas}(K)}{3} \\ &= \textstyle\sum_K \sum_M |\mathbf{grad}\, q_h \cdot \boldsymbol{t}_e|^2 |e|^2 \dfrac{\operatorname{meas}(K)}{3} \\ &\geq C \textstyle\sum_k h_k^2 \|\mathbf{grad}\, q_h\|_{0,K}^2, \end{aligned} \tag{5.27}$$

where in the last inequality we have implicitly used a nondegeneracy condition $|e| \geq \sigma h_K$ and the hypothesis that two sides of K are internal so that $\bar{\boldsymbol{v}}_h$ is defined from $\mathbf{grad}\, q_h \cdot \boldsymbol{t}_e$ in at least two directions on every triangle.

From (5.25) and (5.27) we get

$$\frac{\int_\Omega q_h \operatorname{div} \bar{\boldsymbol{v}}_h \, \mathrm{d}x}{\|\bar{\boldsymbol{v}}_h\|_1} \geq c \frac{\sum_k h_K^2 \|\mathbf{grad}\, q_h\|_{0,K}^2}{\left(\sum_k h_K^2 \|\mathbf{grad}\, q_h\|_{0,K}^2\right)^{1/2}}, \tag{5.28}$$

which is the desired result. □

We can now prove

Proposition 5.3 (Stability of the Taylor–Hood element.) The pair $V_h = (\mathcal{L}_2^1)^2 \cap (H_0^1(\Omega))^2$ and $Q_h = \mathcal{L}_1^1$ is a stable element for the Stokes problem, that is it satisfies the discrete inf–sup condition.

Proof. We multiply the inequality (5.22) by c_3 and (5.23) by c_2 and we add them to get

$$(c_3 + c_2) \sup_{\boldsymbol{v}_h \in V_h} \frac{\int_\Omega q_h \operatorname{div} \boldsymbol{v}_h \, \mathrm{d}x}{\|\boldsymbol{v}_h\|_1} \geq c_1 c_3 \|q_h\|_{0/\mathrm{R}}, \tag{5.29}$$

which is the desired condition. □

This idea of combining a 'bad inequality' -like (5.22) and a 'good inequality in a bad norm' -like (5.23) is due to Verfürth (1984). It can be applied to other situations, for example to the study of stabilized methods presented in the next sections.

6. The Q_1–P_0 element (or 'what might go wrong')

We have introduced, in the previous sections, the discrete inf–sup condition (4.20). It is important in practice to know what should be expected if this condition is *not satisfied.* It is clear that the trouble will arise in the dual problem, that is, with the pressure. The most classical of difficulties is the appearance of a spurious zero-energy mode in the dual problem. All functions in $\ker(\mathbf{grad}_h)$ are zero-energy modes in the dual problem. Those which are nonconstant are known as *chequerboard modes* because of the first discovered case:

Example 6.1 (The Q_1–P_0 element and the chequerboard model.) We consider a Q_1–P_0 approximation, that is we approximate velocity by bilinear elements and pressure by piecewise constants. Moreover, we restrict ourselves to a regular and rectangular mesh. Then, if we colour the rectangles like the squares of a chequerboard, there exists a spurious zero-energy mode taking value 0 on white squares and value 1 on black squares (Figure 6.1).

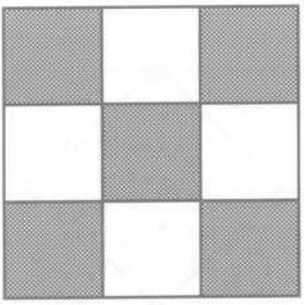

Fig. 6.1. The chequerboard mode.

This mode is defined up to a multiplicative constant and often manifests itself by huge values. In particular, a small displacement of a node by a value of ϵ transforms the zero eigenvalue into an $\mathcal{O}(\epsilon)$ eigenvalue, making an $\mathcal{O}(1/\epsilon)$ chequerboard mode to appear. □

Other examples of zero-energy modes are met in equal-interpolation approximations, that is approximations in which pressure and velocities are approximated by polynomials of the same degree. Most of the time, but not always, they are strongly mesh-dependent and are present only on special regular meshes. The exactly divergence-free element of Example 5.1 on the crossgrid mesh of Figure 5.1 also suffers from exactly the same chequerboard mode as the Q_1–P_0 element.

This, however, is not the only way in which things can go wrong. Another way is that some nonzero eigenvalues become vanishingly small when h decreases, implying that the constant in condition (4.20) is not bounded from below and goes to zero with h. The result is at best a loss in the order of convergence or, worse still, a total loss of convergence. Again, the Q_1–P_0 element provides us with the simplest example. If we consider a regular rectangular mesh and compute the eigenvalues of the dual problem (Malkus, 1981), we see that a large number of them become smaller as h decreases. They can be associated with eigenvectors consisting of a restriction of the chequerboard mode described above to a 2×2 patch of elements. In all cases, a sign of instability first appears in the pressure. It is only in very severe cases that velocities are polluted in a visible way. Derivatives of velocities are however likely to suffer so that computing the vorticity is a good indicator of trouble. To make things still better, it is possible to build special meshes on which the Q_1–P_0 approximation is stable. One of them is presented in the next figure and was introduced by Letallec and Ruas (1986). It is also possible to show that on a regular mesh, formed of 2×2 patches of elements, things are not so bad as would appear from previous considerations: velocity converges at the right order and pressure can be filtered by projecting it on a proper subspace. A proof can be found in Brezzi and Fortin (1991).

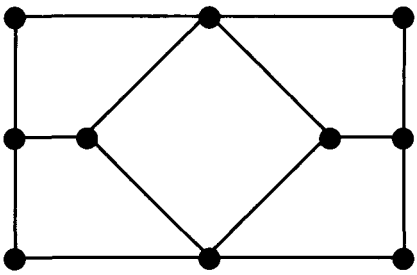

Fig. 6.2. A special mesh for Q_1–P_0.

7. Stabilization techniques

Up to now, we have obtained stable finite element pairs for the approximation of the velocity and the pressure by a clever choice of polynomial spaces. There is, however, another possiblity which has received much attention in recent years: stabilization can be achieved by modifying the variational formulation of the problem. The idea was introduced by Brooks and Hughes (1982) for the stabilization of finite element methods for first-order advection problems. It was later extended in Hughes, Franca and Balestra (1986), Hughes and Franca (1987) and Franca and Hughes (1988) to the Stokes problem, improving on the idea of Brezzi and Pitkäranta (1984) that we shall develop later. Our emphasis will be on the variant of Douglas and Wang (1989) which we consider to be most suitable for the Stokes problem or, more generally, for mixed problems. But let us first consider the formulation of Brezzi and Pitkäranta (1984) which is simple and contains all the basic ideas.

Example 7.1 (The stabilization of Brezzi and Pitkäranta.) The principle is very simple and consists of considering a perturbation of the Stokes problem, that is to modify the problem (4.6) into

$$\begin{cases} a(\boldsymbol{u}_\epsilon, \boldsymbol{v}) + b(\boldsymbol{v}, p_\epsilon) = (\boldsymbol{f}, \boldsymbol{v}), & \forall \boldsymbol{v} \in V, \\ b(\boldsymbol{u}_\epsilon, q) = \epsilon \displaystyle\int_\Omega \mathbf{grad}\, p_\epsilon \cdot \mathbf{grad}\, q \, \mathrm{d}x, & \forall q \in Q. \end{cases} \tag{7.1}$$

This is the variational formulation of the problem,

$$-2\mu\, \boldsymbol{A}\boldsymbol{u}_\epsilon + \mathbf{grad}\, p_\epsilon = \boldsymbol{f}, \tag{7.2}$$

$$\operatorname{div} \boldsymbol{u}_\epsilon + \epsilon \triangle p_\epsilon = 0, \tag{7.3}$$

$$\boldsymbol{u}_\epsilon|_\Gamma = 0, \quad \left.\frac{\partial p_\epsilon}{\partial n}\right|_\Gamma = 0. \tag{7.4}$$

We see from (7.4) that a parasitic Neumann boundary condition has been introduced for the pressure. In practice, this will imply a boundary layer

effect and pressure values will be polluted near the boundary. Nevertheless, one can guess how this model stabilizes an unstable finite element method: chequerboard modes are highly oscillatory and they will be removed by the smoothing effect of the Laplace operator.

The proof that the solution obtained from the method of Example 7.1 is stable will be derived in two steps. First we shall try to obtain a bound on the difference between the solution of the perturbed problem and the solution of the standard Stokes problem. It can be proved (Brezzi and Fortin, 1991) that one has the following estimate.

Proposition 7.1 Let $(\boldsymbol{u}, p)$ tbe the solution of Problem (4.6) and $(\boldsymbol{u}_\epsilon, p_\epsilon)$ be the solution of Problem (7.1). Then we have

$$\|\boldsymbol{u} - \boldsymbol{u}_\epsilon\|_1 + \|p - p_\epsilon\|_0 \leq c\sqrt{\epsilon}\|p\|_1. \tag{7.5}$$

□

We refer to Brezzi and Fortin (1991) for a proof.

This result is not optimal and one can get an $\mathcal{O}(\epsilon^{\frac{3}{2}-\delta})$ estimate if p is smooth enough. However (7.5) is sufficient for our present purpose. Indeed, taking $\epsilon = O(h^2)$ will make the error in (7.5) of the same order as the error in a standard approximation by piecewise linear functions. Therefore, we can discretize Problem (7.1) with the simplest possible elements, such as a P_1–P_1 or a Q_1–Q_1 approximation and obtain results converging with the correct asymptotic accuracy. This gain is, however, not as complete as one would like. The choice of ϵ is critical: if it is too small, pressure oscillation remains while if it is too large, boundary layer effects will spoil the solution. What we would ultimately like to find would be a more robust formulation. A first step toward this is to employ the *Galerkin-least-squares* formulation as in Hughes and Franca (1987). To understand it better, we return to the Lagrangian of Problem (4.7) which we change tentatively to

$$\inf_{\boldsymbol{v}\in V}\sup_{q\in Q} \mu\int_\Omega |\boldsymbol{\varepsilon}(\boldsymbol{v})|^2\,\mathrm{d}x - \int_\Omega q\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x \tag{7.6}$$
$$- \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x - \epsilon\int_\Omega |A\boldsymbol{v} + \mathbf{grad}\,q - \boldsymbol{f}|^2\,\mathrm{d}x.$$

Note that we have added a squared term with a negative sign. This is because we want to stabilize the pressure which is the dual variable in the saddle-point problem. As in the Galerkin-least-squares method, this squared term corresponds to one of the equations in the strong form, namely (7.2). We could have added the square of the second equation to improve the coercivity properties of the problem with respect to $\boldsymbol{u}$. In the present case, this is of no use as the bilinear form $a(\cdot,\cdot)$ is already fully coercive. Examples

of cases where this modification would be useful can be found in Brezzi, Fortin and Marini (1992).

Formulation (7.6) needs a few comments. The first one is that it is in fact ill defined: indeed, for $\boldsymbol{v} \in V$, $A\boldsymbol{v}$ is not square-integrable. We should therefore move to a more regular space, with the side effect of a more difficult approximation, or weaken the formulation as we shall do later. The second comment is that something will go wrong with the coercivity with respect to $\boldsymbol{u}$ as we now have a bilinear form

$$a(\boldsymbol{u}, \boldsymbol{v}) = \mu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \, \mathrm{d}x - \epsilon \int_\Omega A\boldsymbol{v} \cdot A\boldsymbol{v} \, \mathrm{d}x. \tag{7.7}$$

The negative sign impairs the coercivity of $a(\cdot, \cdot)$. Only for discrete problems can this be cured, by taking ϵ small (e.g., $\mathcal{O}(h^2)$), and by using the equivalence of norms on a finite-dimensional space, more precisely an *inverse inequality* of the form

$$\|A\boldsymbol{v}_h\|_0 \leq \frac{C}{h} \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0. \tag{7.8}$$

Let us return to the first point. In order to be able to employ a standard finite element approximation, we shall write the least-squares terms in the form

$$\epsilon \sum_K \int_K |A\boldsymbol{v} + \mathbf{grad}\, q - \boldsymbol{f}|^2 \, \mathrm{d}x. \tag{7.9}$$

This is now well defined on the space

$$W = \{(\boldsymbol{v}, q) | A\boldsymbol{v} + \mathbf{grad}\, q_{|K} \in L^2(K), \forall K \in \mathcal{T}_h\}. \tag{7.10}$$

Standard finite element discretizations of $H^1(\Omega) \times L^2(\Omega)$ are also contained in W as the restriction to an element is a regular polynomial function. This modification does not, however, cure the problem of coercivity. The answer to this second issue is a formulation introduced by Douglas and Wang (1989) where the variational problem (4.6) is modified into

$$\begin{aligned} &a(\boldsymbol{u}_\epsilon, \boldsymbol{v}) + b(\boldsymbol{v}, p_\epsilon) \\ &\quad + \epsilon \textstyle\sum_K \displaystyle\int_K (A\boldsymbol{u}_\epsilon + \mathbf{grad}\, p_\epsilon - \boldsymbol{f}) \cdot A\boldsymbol{v} \, \mathrm{d}x = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in V, \\ &b(\boldsymbol{u}_\epsilon, q) - \epsilon \textstyle\sum_K \displaystyle\int_K (A\boldsymbol{u}_\epsilon + \mathbf{grad}\, p_\epsilon - \boldsymbol{f}) \cdot \mathbf{grad}\, q \, \mathrm{d}x = 0, \quad \forall q \in Q. \end{aligned} \tag{7.11}$$

This differs by one sign change from what would be obtained by the optimality conditions of Problem (7.6). This sign change is nevertheless crucial: choosing $\boldsymbol{v} = \boldsymbol{u}_\epsilon$ and $q = p_\epsilon$ in (7.11) and substracting the two equations one gets

$$\mu \int_\Omega |\boldsymbol{\varepsilon}(\boldsymbol{u})_\epsilon|^2 \, \mathrm{d}x + \epsilon \sum_K \int_K |A\boldsymbol{u}_\epsilon + \mathbf{grad}\, p_\epsilon|^2 \, \mathrm{d}x \leq c(\|\boldsymbol{f}\|^2). \tag{7.12}$$

We have therefore obtained stability in the space W defined in (7.10) for any value of ϵ. It might appear that we have not really stabilized $\mathbf{grad}\, p$ but only $A\boldsymbol{u} + \mathbf{grad}\, p$. However, as $\boldsymbol{u}$ is generally smooth, the stabilization effort really bears on p. Formulations of this type have been used with success with many finite element formulations. They allow equal-interpolation approximations of low order which would otherwise be forbidden in standard variational formulations. Everything is not as nice as it would seem from the above discussion: the solution still suffers from a parasitic boundary condition on p and a boundary layer effect. The source of trouble is that the term $A\boldsymbol{u}$ in $A\boldsymbol{u} + \mathbf{grad}\, p - \boldsymbol{f}$ is not computed accurately in a standard finite element approximation. It will normally be approximated at one order lower than the other terms. The limiting case is the piecewise linear one where $A\boldsymbol{u}|_K$ is always identically zero. This will oblige us to take again $\epsilon = \mathcal{O}(h^2)$ to recover the corrrect error estimate. Moreover, this lack of accuracy in one term spoils the solution in a visible way near the boundary. Many techniques have been advocated to remove this boundary layer effect (e.g., Brezzi and Douglas (1988)). The most popular one consists in substracting boundary effects by adding a correcting term to the formulation. For example, one might modify Problem (7.11) into

$$\begin{cases} a(\boldsymbol{u}_\epsilon, \boldsymbol{v}) + b(\boldsymbol{v}, p_\epsilon) \\ \qquad + \epsilon \sum_K \int_K (A\boldsymbol{u}_\epsilon + \mathbf{grad}\, p_\epsilon - \boldsymbol{f}) \cdot A\boldsymbol{v}\, \mathrm{d}x = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in V, \\ b(\boldsymbol{u}_\epsilon, q) - \epsilon \sum_K \int_K (A\boldsymbol{u}_\epsilon + \mathbf{grad}\, p_\epsilon - \boldsymbol{f}) \cdot \mathbf{grad}\, q\, \mathrm{d}x \\ \qquad - \int_{\partial\Omega} ((A\boldsymbol{u}_\epsilon + \mathbf{grad}\, p_\epsilon - \boldsymbol{f}) \cdot \boldsymbol{n}\, q\, \mathrm{d}s = 0, \quad \forall q \in Q. \end{cases} \tag{7.13}$$

Numerical results obtained through such modifications are good (cf., e.g., Leborgne (1992)). However, coercivity properties are lost and getting a solution from the discretized problem becomes delicate. The correct way of eliminating boundary layer effects is still an open problem. To conclude, stabilized formulations are an important new idea in the approximation of incompressility, an idea which is likely to see new developments in future years.

8. Numerical methods for the discretized problems

Given a stable approximation, we now have the practical task of effectively computing the approximate solution. We shall deal with two different issues, namely the treatment of the incompressibility condition and the treatment of the full nonlinear Navier–Stokes problem.

8.1. Penalty methods

The numerical methods which we want to introduce are based mainly on techniques derived from penalty methods. We shall therefore describe these briefly. We use the steady-state Stokes problem as a prototype but the idea applies to any incompressible problem. A penalty method is then a perturbation of our original problem (4.6) into

$$\begin{cases} a(\boldsymbol{u}_\epsilon, \boldsymbol{v}) + b(\boldsymbol{v}, p_\epsilon) = (\boldsymbol{f}, \boldsymbol{v}), & \forall \boldsymbol{v} \in V, \\ b(\boldsymbol{u}_\epsilon, q) - \epsilon \int_\Omega p_\epsilon q \, \mathrm{d}x = 0, & \forall q \in Q. \end{cases} \tag{8.1}$$

It can be shown (Bercovier (1978) or Brezzi and Fortin (1991)) that the error induced by this (regular) perturbation is $\mathcal{O}(\epsilon)$. Let us now consider the matrix form of the discrete problem already presented in Section(4.5). The problem becomes

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^t & -\epsilon \mathcal{M}^Q \end{pmatrix} \begin{pmatrix} \boldsymbol{U} \\ \boldsymbol{P} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F} \\ \mathbf{o} \end{pmatrix}. \tag{8.2}$$

But the matrix $\mathcal{M}^Q$ is invertible and it is possible to eliminate pressure from these equations to obtain

$$\mathcal{A}\boldsymbol{U} + \frac{1}{\epsilon}\mathcal{B}^t(\mathcal{M}^Q)^{-1}\mathcal{B}\boldsymbol{U} = \boldsymbol{F}. \tag{8.3}$$

Once $\boldsymbol{U}$ has been obtained by solving (8.3), one can calculate the pressure by

$$\boldsymbol{P} = \frac{1}{\epsilon}(\mathcal{M}^Q)^{-1}\mathcal{B}\boldsymbol{U}. \tag{8.4}$$

This procedure is in fact usable only if the matrix $\mathcal{M}^Q$ is easily invertible (Bercovier, Engelman and Gresho (1982)). For discontinuous pressure approximations described in Section 5, we can invert $\mathcal{M}^Q$ element by element and the numerical implementation is direct. It must be said that this simplification also has some disadvantages: the system (8.3) is ill conditioned for ϵ small. Care must be taken if one wishes to get an accurate solution, and the convergence of iterative methods, such as a conjugate-gradient method is jeopardized. For continuous pressure approximations, $(\mathcal{M}^Q)^{-1}$ is a full matrix and the reduced problem is not tractable. The perturbed problem (8.2) is nevertheless employed as it cures the singularity (p is defined up to an additive constant) of the original problem in the case of pure Dirichlet conditions on $\boldsymbol{u}$.

8.2. The augmented Lagrangian method

We briefly describe here how a simple iterative procedure, called the augmented Lagrangian method, can be employed to remove penalty errors and

to efficiently compute a solution of the original problem (4.16). Our presentation will be sketchy by necessity. We refer to Fortin and Glowinski (1983) for a precise analysis of the method. This technique is also closely related to the artificial compressibility method introduced by Chorin (1968) and widely used under different names.

Suppose that we choose $\boldsymbol{P}_0$, an arbitrary initial guess for the pressure. We then compute, $\boldsymbol{P}_n$ being known, $\boldsymbol{P}_{n+1}$ from the relation,

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^t & -\epsilon\mathcal{M}^Q \end{pmatrix} \begin{pmatrix} \boldsymbol{U}_{n+1} \\ \boldsymbol{P}_{n+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F} \\ -\epsilon\mathcal{M}^Q\boldsymbol{P}_n \end{pmatrix}. \tag{8.5}$$

If $\mathcal{M}^Q$ is easily invertible, one can write this in the decoupled form,

$$\mathcal{A}\boldsymbol{U}_n + \frac{1}{\epsilon}\mathcal{B}^t(\mathcal{M}^Q)^{-1}\mathcal{B}\boldsymbol{U}_n = \boldsymbol{F} - \mathcal{B}^t\boldsymbol{P}_n, \tag{8.6}$$

$$\mathcal{M}^Q\boldsymbol{P}_{n+1} = \mathcal{M}^Q\boldsymbol{P}_n + \frac{1}{\epsilon}\mathcal{B}\boldsymbol{U}_n. \tag{8.7}$$

This is a special case of a more general algorithm, Uzawa's algorithm, for the numerical solution of saddle-point problems. Convergence is easily proved for any positive value of ϵ. Taking ϵ small, (say 10^{-6}) makes the algorithm convergent to machine precision in two or three iterations. In fact taking ϵ small makes the dual problem in p very well conditioned (cf. Fortin and Pierre (1992)) so that this iteration, which is in fact a gradient method for the dual problem, converges very rapidly. The price we pay is that Problem (8.6) in $\boldsymbol{U}$ is ill conditioned. When an iterative solution is needed, as is often the case in three-dimensional problems, a balance should be kept between the convergence of the iteration for solving Problem (8.6) and the convergence of the outer iteration in (8.7). Methods of this type have also been used as preconditioners for conjugate-gradient methods (e.g., Fortin (1989)).

8.3. Nonlinear problems

When the Navier–Stokes problem is considered, we have to solve a large nonlinear system. The most popular method is Newton's method which reduces this solution to a sequence of linear incompressible problems. The augmented Lagrangian method can then be used to solve these linear problems. Under some restrictions on the choice of ϵ it can also be incorporated to Newton's iteration (Fortin and Fortin, 1985b). The most efficient solution method is, however, to employ a conjugate-gradient-like iteration such as the GMRES method of Saad and Schultz (1986) with a suitable preconditioning. One then needs only to compute products of some vectors and the Jacobian matrix and this can be approximated by differences, avoiding the actual computation of the Jacobian. A very good description of this technique can be found in Shakib, Hughes and Zdeněk (1989) for compressible

problems and can be transposed directly to incompressible problems. For incompressible problems, some approximate augmented Lagrangian method can be used as a preconditioner (Fortin, 1989).

9. Time-dependent problems

Our original problem was time-dependent and we now return to this aspect. The standard procedure for the discretization of a time-dependent problem is to first consider a discretization in space, reducing it to a large system of ordinary differential equations and then to employ some numerical scheme for this system. The choice of scheme can then be made from a vast collection of ODE solvers.

9.1. Time discretization, projection methods

One important point in the choice of a time discretization is that the system is not of Cauchy–Kovalevska type as there is no derivative in time of the pressure in the equations. In fact, in this respect, the problem is related to the so-called algebraic–differential systems (Petzold, 1983). It can be seen that the pressure part is elliptic. Indeed, taking the divergence of equation (3.1), we obtain the *Poisson pressure equation*

$$-\triangle p = \operatorname{div}(\boldsymbol{u}\cdot\mathbf{grad}\,\boldsymbol{u}) - \operatorname{div}\boldsymbol{f}, \tag{9.1}$$

in which we have used (3.2) to eliminate a number of terms. This equation (9.1) holds at all times. It has been widely employed in the construction of time-stepping procedures, but difficulties arise from the absence of pressure boundary conditions. There is, in reality, no rigorous way to obtain such conditions apart from some iterative procedure or the construction of an integral equation on the boundary of the domain like in Glowinski and Pironneau (1979) (see also Gresho and Sani (1987)). This being said, it is possible to include the solution of a Neumann problem in p into a fractional step method, such as the projection method introduced in Chorin (1968) and developed in Fortin, Peyret and Temam (1971). This scheme, in its simplest form, would consists of an advection step followed by a projection on the subspace of divergence-free functions. However, it is not immediately clear in which space should the projection take place. The two obvious choices are $L^2(\Omega)$ or $H_0^1(\Omega)$. Let us consider in some detail these two cases.

Lemma 9.1 (The $L^2(\Omega)$ projection.) Let $\boldsymbol{u}$ be given in $(L^2(\Omega))^n$. Then $\boldsymbol{u}$ can be written as

$$\boldsymbol{u} = \boldsymbol{u}_0 + \mathbf{grad}\,p_0 + \mathbf{grad}\,p_1, \tag{9.2}$$

with $p_0 \in H_0^1(\Omega)$, $p_1 \in \mathcal{H}(\Omega)$ and $\boldsymbol{u}_0 \in H_0(\Omega)$, where

$$\begin{cases} \mathcal{H}(\Omega) = \{q \mid q \in H^1(\Omega), \triangle p = 0\}, \\ H_0(\Omega) = \{\boldsymbol{v} \mid \boldsymbol{v} \in (L^2(\Omega))^n, \operatorname{div}\boldsymbol{v} = 0, \boldsymbol{v}\cdot\boldsymbol{n}_{|\partial\Omega} = 0\}. \end{cases} \tag{9.3}$$

Proof. The idea is essentially to solve a Dirichlet problem,

$$-\triangle p_0 = \operatorname{div} \boldsymbol{u}, \tag{9.4}$$

to compute $\boldsymbol{u}_1 = \boldsymbol{u} - \mathbf{grad}\, p_0$, to solve a Neumann problem

$$-\triangle p_1 = 0, \quad \frac{\partial p_1}{\partial \boldsymbol{n}} = \boldsymbol{u}_1 \cdot \boldsymbol{n} \text{ on } \partial\Omega, \tag{9.5}$$

and then finally obtain $\boldsymbol{u}_0 = \boldsymbol{u}_1 - \mathbf{grad}\, p_1$. □

A few remarks are needed about the boundary conditions employed. First, the condition $\boldsymbol{u}_0 \cdot \boldsymbol{n}$ has to be justified for it does not make sense, *a priori*, to write a boundary condition for a function in $(L^2(\Omega))^n$. We refer to Temam (1977) for this justification. Second, the two problems (9.4)–(9.5) may be combined into one:

$$-\triangle p = \operatorname{div} \boldsymbol{u}, \quad \frac{\partial p}{\partial \boldsymbol{n}} = \boldsymbol{u} \cdot \boldsymbol{n} \text{ on } \partial\Omega, \tag{9.6}$$

provided $\boldsymbol{u}$ slightly more regular, namely if $\operatorname{div} \boldsymbol{u} \in L^2(\Omega)$. Let us now see how one can use this approach to construct a fractional step method.

Example 9.1 (The L^2 projection scheme.) This scheme will be a fractional step method and many variants are possible, depending on the implicit or explicit character of the first step. We shall consider here the implicit variant, which we feel is more reliable but other cases can be easily formulated. Moreover, we shall not explicitly introduce a space discretization and we shall, formally, write the scheme without any such discretization. Let then $\boldsymbol{u}^n$ and p^n be known at time step n. We shall compute the solution at the next time step $n+1$ in two substeps. First, we solve, denoting by δt the time step,

$$\frac{\boldsymbol{u}^{n+\frac{1}{2}} - \boldsymbol{u}^n}{\delta t} + \boldsymbol{u}^{n+\frac{1}{2}} \cdot \mathbf{grad}\, \boldsymbol{u}^{n+\frac{1}{2}} - 2\mu\, \boldsymbol{A}\boldsymbol{u}^{n+\frac{1}{2}} + \mathbf{grad}\, p^n = \boldsymbol{f}. \tag{9.7}$$

This is a nonlinear problem which can be solved either by a Newton method or an approximate Newton method. No incompressibility condition is imposed on $\boldsymbol{u}^{n+\frac{1}{2}}$ and the next step intends to correct this deficiency by projecting it on the divergence-free subspace $H_0(\Omega)$. This amounts to solving a Neumann problem:

$$-\triangle \delta p = \operatorname{div} \boldsymbol{u}^{n+\frac{1}{2}}, \quad \frac{\partial \delta p}{\partial \boldsymbol{n}} = \boldsymbol{u}^{n+\frac{1}{2}} \cdot \boldsymbol{n} \text{ on } \partial\Omega \tag{9.8}$$

and then to compute

$$\begin{cases} \boldsymbol{u}^{n+1} = \boldsymbol{u}^{n+\frac{1}{2}} - \mathbf{grad}\, \delta p, \\ p^{n+1} = p^n + \delta p. \end{cases} \tag{9.9}$$

This is simple, but something is wrong: $\boldsymbol{u}^{n+1}$ does not satisfy the correct boundary conditions because the projection step only requires $\boldsymbol{u}^{n+1} \cdot \boldsymbol{n}$ to be null and leaves the tangential condition $\boldsymbol{u}^{n+1} \cdot \boldsymbol{t}$ undetermined. In practice, this problem is bypassed, in a discretized setting, by inserting the correct values at boundary nodes after the projection step. This is a new projection step, in some nonexplicit topology. The result is a scheme which is essentially first order in δt. □

To do improve this, we would like to be able to project in $(H_0^1(\Omega))^n$-norm. But this is essentially equivalent to solving a Stokes problem.

Lemma 9.2 (The $(H_0^1(\Omega))^n$-projection.) Let $\boldsymbol{u}$ be given in $(H_0^1(\Omega)^n$. Then $\boldsymbol{u}$ can be written as

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_p, \tag{9.10}$$

where $\boldsymbol{u}_0 \in V_0(\Omega)$ with

$$V_0(\Omega) = \{\boldsymbol{v} \mid \boldsymbol{v} \in (H_0^1(\Omega)^n, \operatorname{div} \boldsymbol{v} = 0\}, \tag{9.11}$$

and is the solution of the problem

$$\begin{cases} \boldsymbol{A}\boldsymbol{u}_0 + \mathbf{grad}\, m = \boldsymbol{A}\boldsymbol{u}, \\ \operatorname{div} \boldsymbol{u}_0 = 0, \end{cases} \tag{9.12}$$

where m is analogous to a pressure and serves to enforce the divergence-free condition.

Proof. The problem is to find $\boldsymbol{u}_0$ as the solution of the constrained minimization problem,

$$\inf_{\boldsymbol{u}_0 \in V_0} \|\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \boldsymbol{\varepsilon}(\boldsymbol{u})\|_{1,\Omega}^2. \tag{9.13}$$

Introducing the Lagrange multiplier m and writing the optimality conditions of the Lagrangian obtained, one gets (9.12). □

Using this result we are naturally led to a new projection scheme.

Example 9.2 (The H_0^1-projection scheme.) Let $\boldsymbol{u}^n$ and p^n be known at time step n. We shall compute the solution at the next time step $n+1$ in two substeps. First, we solve, denoting by δt the time step,

$$\frac{\boldsymbol{u}^{n+\frac{1}{2}} - \boldsymbol{u}^n}{\delta t} + \boldsymbol{u}^{n+\frac{1}{2}} \cdot \mathbf{grad}\, \boldsymbol{u}^{n+\frac{1}{2}} - 2\mu\, \boldsymbol{A}\boldsymbol{u}^{n+\frac{1}{2}} + \mathbf{grad}\, p^n = \boldsymbol{f}. \tag{9.14}$$

We then project $\boldsymbol{u}^{n+\frac{1}{2}}$ by solving

$$\begin{cases} \boldsymbol{A}\boldsymbol{u}^{n+1} + \mathbf{grad}\, \delta p = \boldsymbol{A}\boldsymbol{u}^{n+\frac{1}{2}}, \\ \operatorname{div} \boldsymbol{u}_{n+1} = 0, \\ p^{n+1} = p^n + \delta p. \end{cases} \tag{9.15}$$

This is only one of possible variants and second-order methods can also be built using the θ-scheme such as in Bristeau, Glowinski and Périaux (198?) or other techniques (Bell, Colella and Glaz, 1989). The price we pay for this better handling of boundary conditions in the projection step is that the Stokes Problem (9.15) defining $\boldsymbol{u}^{n+1}$ is harder to solve than the Neumann problem (9.15). Using this method means that one should dispose of an efficient and simple Stokes solver. □

Finally, a brute force method, that is a fully implicit scheme, can also be employed.

Example 9.3 (The fully implicit scheme.) Let $\boldsymbol{u}^n$ and p^n be known at time step n. We compute the solution at the next time step $n+1$ by solving:

$$\begin{cases} \dfrac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{\delta t} + \boldsymbol{u}^{n+1}\cdot\mathbf{grad}\,\boldsymbol{u}^{n+1} - 2\mu\,\boldsymbol{A}\boldsymbol{u}^{n+1} + \mathbf{grad}\,p^{n+1} = \boldsymbol{f}, \\ \\ \operatorname{div}\boldsymbol{u}^{n+1} = 0. \end{cases} \tag{9.16}$$

Now $\boldsymbol{u}^{n+1}$ is the solution of a nonlinear incompressible problem. One possible way to solve this is by the penalty method already discussed. An equivalent way of introducing it is through the 'artificial compressibility method' of Chorin (1968) which is usually written as a perturbation of the above scheme:

$$\begin{cases} \dfrac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{\delta t} + \boldsymbol{u}^{n+1}\cdot\mathbf{grad}\,\boldsymbol{u}^{n+1} - 2\mu\,\boldsymbol{A}\boldsymbol{u}^{n+1} + \mathbf{grad}\,p^{n+1} = \boldsymbol{f}, \\ \\ \epsilon\dfrac{p^{n+1}-p^n}{\delta t} - \operatorname{div}\boldsymbol{u}^{n+1} = 0. \end{cases} \tag{9.17}$$

Using the second equation, the first one may be written as

$$\begin{aligned} &\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{\delta t} + \boldsymbol{u}^{n+1}\cdot\mathbf{grad}\,\boldsymbol{u}^{n+1} \\ &\qquad -2\mu\,\boldsymbol{A}\boldsymbol{u}^{n+1} + \frac{\delta t}{\epsilon}\mathbf{grad}\operatorname{div}\boldsymbol{u}^{n+1} + \mathbf{grad}\,p^n = \boldsymbol{f}, \end{aligned} \tag{9.18}$$

which is nothing but a penalty method for the solution of (9.16). One also sees that ϵ should be small with respect to δt which may give rise to severe ill-conditioning. An iterative variant based on the augmented Lagrangian method is therefore much more preferable. □

Remark 9.1 In the implicit scheme (9.16) we have used an implicit Euler's scheme which is a stiffly-stable implicit method for ordinary differential equations (cf. Crouzeix and Mignot (1984). This strong stability property is highly desirable for large systems. However, it is now quite well established

that to detect bifurcations to unsteady solutions in nonlinear Navier–Stokes problems correctly, a second-order scheme is essential (Fortin, Fortin and Gervais, 1991). A reasonable solution is through Gear's method which is a two-step implicit stiffly-stable scheme. It requires knowledge of $\boldsymbol{u}^n$ and $\boldsymbol{u}^{n-1}$ to compute $\boldsymbol{u}^{n+1}$.

$$\begin{cases} \dfrac{\frac{3}{2}\boldsymbol{u}^{n+1} - 2\boldsymbol{u}^n + \frac{1}{2}\boldsymbol{u}^{n-1}}{\delta t} + \boldsymbol{u}^{n+1}\cdot\mathbf{grad}\,\boldsymbol{u}^{n+1} \\ \qquad\qquad -2\mu\,\boldsymbol{A}\boldsymbol{u}^{n+1} + \mathbf{grad}\,p^{n+1} = \boldsymbol{f}, \\ \operatorname{div}\boldsymbol{u}^{n+1} = 0. \end{cases} \tag{9.19}$$

This scheme has been successfully employed for the computation of Hopf bifurcations. □

Remark 9.2 An interesting variant for a totally implicit scheme consists in using a method of characteristics for the discretization of advection terms (cf. Pironneau (1989)). The simplest way to do so can be summarized in the following algorithm.

$$\text{For any vertex } V \text{ of coordinates } \boldsymbol{x} \text{ compute } \boldsymbol{x}_* = \boldsymbol{x} - \delta t \tag{9.20}$$

$$\text{Compute } \boldsymbol{u}_*(\boldsymbol{x}, t_n) = \boldsymbol{u}(\boldsymbol{x}_*, t_n). \tag{9.21}$$

To compute $\boldsymbol{u}^{n+1}$ one then solves

$$\begin{cases} \dfrac{\boldsymbol{u}^{n+1} - \boldsymbol{u}_*}{\delta t} - 2\mu\,\boldsymbol{A}\boldsymbol{u}^{n+1} + \mathbf{grad}\,p^{n+1} = \boldsymbol{f}, \\ \operatorname{div}\boldsymbol{u}^{n+1} = 0. \end{cases} \tag{9.22}$$

The problem to solve in $\boldsymbol{u}^{n+1}$ is then a linear problem which can be solved by any suitable Stokes solver. More sophisticated versions of this idea are currently employed in industrial codes.

10. Conclusion

The possible issues to be considered in the numerical solution of the Navier–Stokes equations are so numerous that only a small fraction of them has been addressed here. Some, such as solution algorithms, have only been sketched. Finally, questions related to *a posteriori* error estimations and adaptivity have been completely ignored. The main difficulty remaining in the field is certainly the treatment of flows at high Reynolds number. Boundary layers imply delicate questions of mesh adaptation. Turbulence models, which try to represent the macroscopic effects of the small scales of the flow, are also an important issue. With respect to the treatment of incompressibilty which was our main topic, three-dimensional problems remain a challenge in both the construction of accurate elements and in the design of efficient solution

methods. We hope that this article shall be useful as a guide into the rapidly changing world of computational fluid dynamics.

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